Section II: Geometry

7 GEOMETRIC FORMULAS

RECTANGLE OF LENGTH b AND WIDTH a

7.1 Area $= ab$

7.2 Perimeter $= 2a + 2b$

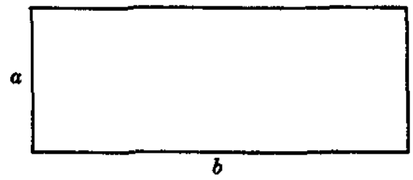

Fig. 7-1

PARALLELOGRAM OF ALTITUDE h AND BASE b

7.3 Area $= bh = ab \sin\theta$

7.4 Perimeter $= 2a + 2b$

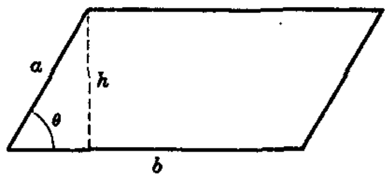

Fig. 7-2

TRIANGLE OF ALTITUDE h AND BASE b

7.5 Area $= \frac{1}{2}bh = \frac{1}{2}ab\sin\theta$

$= \sqrt{s(s-a)(s-b)(s-c)}$

where $s = \frac{1}{2}(a+b+c) =$ semiperimeter

7.6 Perimeter $= a + b + c$

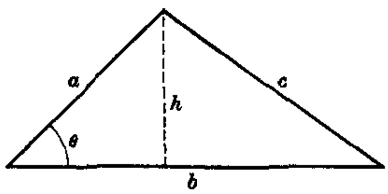

Fig. 7-3

TRAPEZOID OF ALTITUDE h AND PARALLEL SIDES a AND b

7.7 Area $= \frac{1}{2}h(a+b)$

7.8 Perimeter $= a + b + h\left(\frac{1}{\sin\theta} + \frac{1}{\sin\phi}\right)$

$= a + b + h(\csc\theta + \csc\phi)$

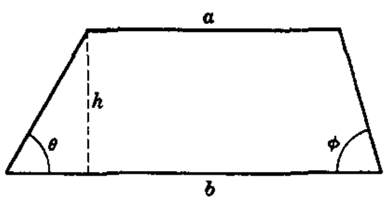

Fig. 7-4

REGULAR POLYGON OF n SIDES EACH OF LENGTH b

7.9 Area $= \frac{1}{4}nb^2 \cot\frac{\pi}{n} = \frac{1}{4}nb^2 \frac{\cos(\pi/n)}{\sin(\pi/n)}$

7.10 Perimeter $= nb$

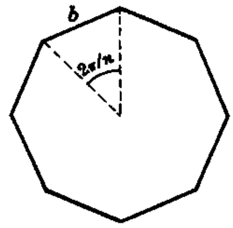

Fig. 7-5

CIRCLE OF RADIUS r

7.11 Area $= \pi r^2$

7.12 Perimeter $= 2\pi r$

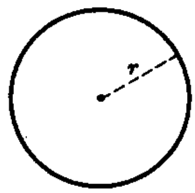

Fig. 7-6

SECTOR OF CIRCLE OF RADIUS r

7.13 Area $= \frac{1}{2}r^2\theta$ [θ in radians]

7.14 Arc length $s = r\theta$

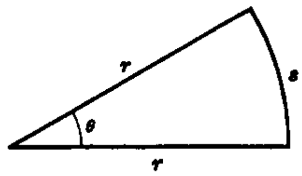

Fig. 7-7

RADIUS OF CIRCLE INSCRIBED IN A TRIANGLE OF SIDES a, b, c

7.15 $$r = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}$$

where $s = \frac{1}{2}(a+b+c)$ = semiperimeter.

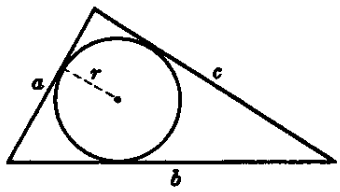

Fig. 7-8

RADIUS OF CIRCLE CIRCUMSCRIBING A TRIANGLE OF SIDES a, b, c

7.16 $$R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$$

where $s = \frac{1}{2}(a+b+c)$ = semiperimeter.

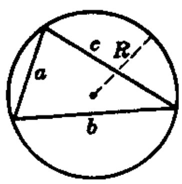

Fig. 7-9

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REGULAR POLYGON OF n SIDES INSCRIBED IN CIRCLE OF RADIUS r

7.17 Area $= \frac{1}{2}nr^2 \sin\frac{2\pi}{n} = \frac{1}{2}nr^2 \sin\frac{360^\circ}{n}$

7.18 Perimeter $= 2nr\sin\frac{\pi}{n} = 2nr\sin\frac{180^\circ}{n}$

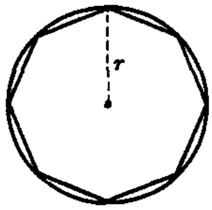

Fig. 7-10

REGULAR POLYGON OF n SIDES CIRCUMSCRIBING A CIRCLE OF RADIUS r

7.19 Area $= nr^2\tan\frac{\pi}{n} = nr^2\tan\frac{180^\circ}{n}$

7.20 Perimeter $= 2nr\tan\frac{\pi}{n} = 2nr\tan\frac{180^\circ}{n}$

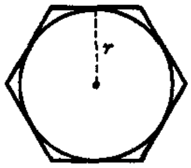

Fig. 7-11

SEGMENT OF CIRCLE OF RADIUS r

7.21 Area of shaded part $= \frac{1}{2}r^2(\theta - \sin\theta)$

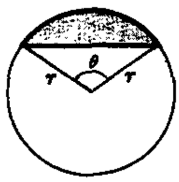

Fig. 7-12

ELLIPSE OF SEMI-MAJOR AXIS a AND SEMI-MINOR AXIS b

7.22 Area $= \pi ab$

7.23 Perimeter $= 4a\int_0^{\pi/2}\sqrt{1-k^2\sin^2\theta}\,d\theta$

$= 2\pi\sqrt{\frac{1}{2}(a^2+b^2)}$ [approximately]

where $k = \sqrt{a^2-b^2}/a$. See Table 43 for numerical values.

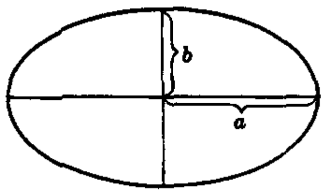

Fig. 7-13

SEGMENT OF A PARABOLA

7.24 Area $= \frac{2}{3}ab$

7.25 Arc length $ABC = \frac{1}{2}\sqrt{b^2+16a^2} + \frac{b^2}{8a}\ln\left(\frac{4a+\sqrt{b^2+16a^2}}{b}\right)$

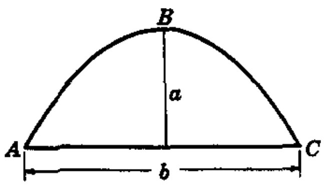

Fig. 7-14

RECTANGULAR PARALLELEPIPED OF LENGTH a, HEIGHT l, WIDTH c

7.26 Volume = abc

7.27 Surface area = $2(ab + ac + bc)$

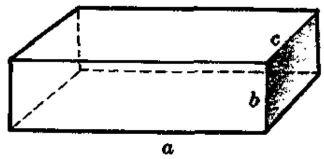

Fig. 7-15

PARALLELEPIPED OF CROSS-SECTIONAL AREA A AND HEIGHT h

7.28 Volume = $Ah = abc \sin\theta$

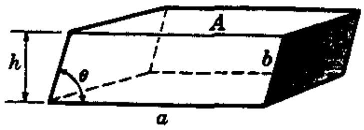

Fig. 7-16

SPHERE OF RADIUS r

7.29 Volume = $\frac{4}{3}\pi r^3$

7.30 Surface area = $4\pi r^2$

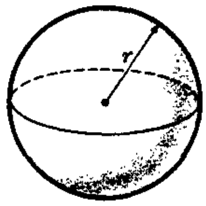

Fig. 7-17

RIGHT CIRCULAR CYLINDER OF RADIUS r AND HEIGHT h

7.31 Volume = $\pi r^2 h$

7.32 Lateral surface area = $2\pi rh$

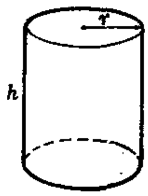

Fig. 7-18

CIRCULAR CYLINDER OF RADIUS r AND SLANT HEIGHT l

7.33 Volume = $\pi r^2 h = \pi r^2 l \sin\theta$

7.34 Lateral surface area = $2\pi rl = \frac{2\pi rh}{\sin\theta} = 2\pi rh \csc\theta$

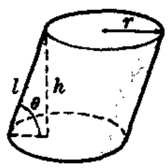

Fig. 7-19

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CYLINDER OF CROSS-SECTIONAL AREA A AND SLANT HEIGHT l

7.35 Volume $= Ah = Al \sin\theta$

7.36 Lateral surface area $= ph = pl \sin\theta$

Note that formulas 7.31 to 7.34 are special cases of formulas 7.35 and 7.36.

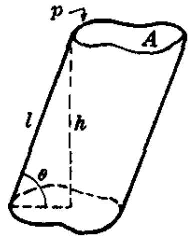

Fig. 7-20

RIGHT CIRCULAR CONE OF RADIUS r AND HEIGHT h

7.37 Volume $= \frac{1}{3}\pi r^2 h$

7.38 Lateral surface area $= \pi r\sqrt{r^2 + h^2} = \pi r l$

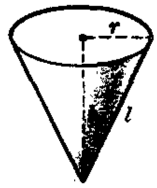

Fig. 7-21

PYRAMID OF BASE AREA A AND HEIGHT h

7.39 Volume $= \frac{1}{3}Ah$

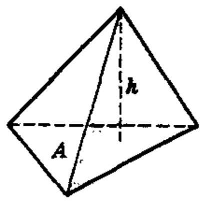

Fig. 7-22

SPHERICAL CAP OF RADIUS r AND HEIGHT h

7.40 Volume (shaded in figure) $= \frac{1}{3}\pi h^2(3r - h)$

7.41 Surface area $= 2\pi r h$

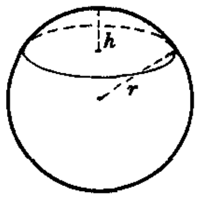

Fig. 7-23

FRUSTUM OF RIGHT CIRCULAR CONE OF RADII a, b AND HEIGHT h

7.42 Volume $= \frac{1}{3}\pi h(a^2 + ab + b^2)$

7.43 Lateral surface area $= \pi(a + b)\sqrt{h^2 + (b - a)^2}$
$= \pi(a + b)l$

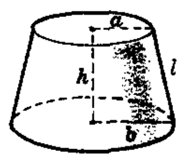

Fig. 7-24

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SPHERICAL TRIANGLE OF ANGLES A, B, C ON SPHERE OF RADIUS r

7.44 Area of triangle $ABC = (A + B + C - \pi)r^2$

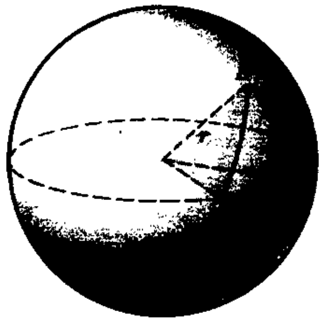

Fig. 7-25

TORUS OF INNER RADIUS a AND OUTER RADIUS b

7.45 Volume $= \frac{1}{4}\pi^2(a + b)(b - a)^2$

7.46 Surface area $= \pi^2(b^2 - a^2)$

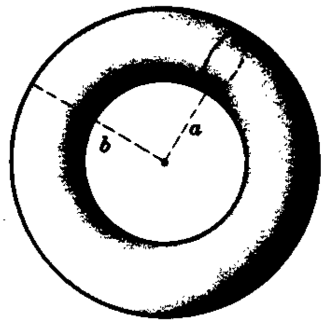

Fig. 7-26

ELLIPSOID OF SEMI-AXES a, b, c

7.47 Volume $= \frac{4}{3}\pi abc$

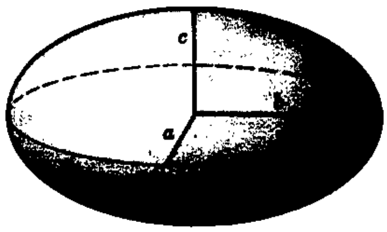

Fig. 7-27

PARABOLOID OF REVOLUTION

7.48 Volume $= \frac{1}{2}\pi b^2 a$

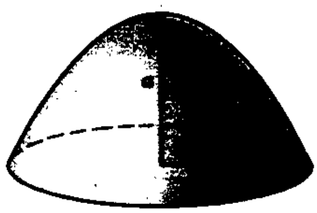

Fig. 7-28

8 FORMULAS from PLANE ANALYTIC GEOMETRY

DISTANCE d BETWEEN TWO POINTS $P_1(x_1, y_1)$ AND $P_2(x_2, y_2)$

8.1 $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

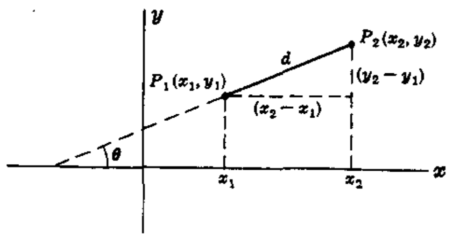

Fig. 8-1

SLOPE m OF LINE JOINING TWO POINTS $P_1(x_1, y_1)$ AND $P_2(x_2, y_2)$

8.2 $$m = \frac{y_2 - y_1}{x_2 - x_1} = \tan\theta$$

EQUATION OF LINE JOINING TWO POINTS $P_1(x_1, y_1)$ AND $P_2(x_2, y_2)$

8.3 $$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} = m \quad \text{or} \quad y - y_1 = m(x - x_1)$$

8.4 $$y = mx + b$$

where $b = y_1 - mx_1 = \dfrac{x_2 y_1 - x_1 y_2}{x_2 - x_1}$ is the *intercept* on the y axis, i.e. the y *intercept*.

EQUATION OF LINE IN TERMS OF x INTERCEPT $a \neq 0$ AND y INTERCEPT $b \neq 0$

8.5 $$\frac{x}{a} + \frac{y}{b} = 1$$

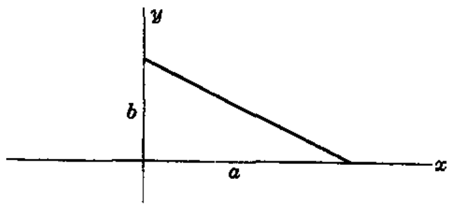

Fig. 8-2

NORMAL FORM FOR EQUATION OF LINE

8.6 $$x \cos \alpha + y \sin \alpha = p$$

where p = perpendicular distance from origin O to line

and α = angle of inclination of perpendicular with positive x axis.

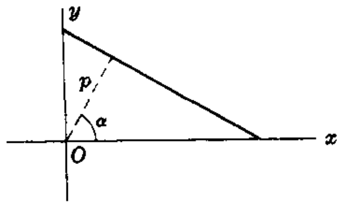

Fig. 8-3

GENERAL EQUATION OF LINE

8.7 $$Ax + By + C = 0$$

DISTANCE FROM POINT (x_1, y_1) TO LINE $Ax + By + C = 0$

8.8 $$\frac{Ax_1 + By_1 + C}{\pm\sqrt{A^2 + B^2}}$$

where the sign is chosen so that the distance is nonnegative.

ANGLE ψ BETWEEN TWO LINES HAVING SLOPES m_1 AND m_2

8.9 $$\tan \psi = \frac{m_2 - m_1}{1 + m_1 m_2}$$

Lines are parallel or coincident if and only if $m_1 = m_2$.
Lines are perpendicular if and only if $m_2 = -1/m_1$.

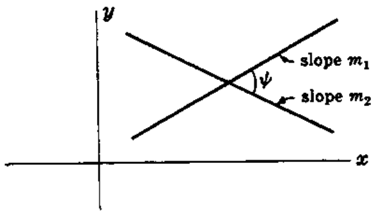

Fig. 8-4

AREA OF TRIANGLE WITH VERTICES AT (x_1, y_1), (x_2, y_2), (x_3, y_3)

8.10 $$\text{Area} = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$= \pm\frac{1}{2}(x_1y_2 + y_1x_3 + y_3x_2 - y_2x_3 - y_1x_2 - x_1y_3)$$

where the sign is chosen so that the area is nonnegative.
If the area is zero the points all lie on a line.

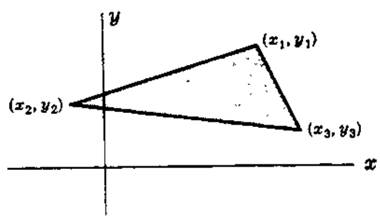

Fig. 8-5

TRANSFORMATION OF COORDINATES INVOLVING PURE TRANSLATION

8.11 $\begin{cases} x = x' + x_0 \\ y = y' + y_0 \end{cases}$ or $\begin{cases} x' = x - x_0 \\ y' = y - y_0 \end{cases}$

where (x, y) are old coordinates [i.e. coordinates relative to xy system], (x', y') are new coordinates [relative to $x'y'$ system], and (x_0, y_0) are the coordinates of the new origin O' relative to the old xy coordinate system.

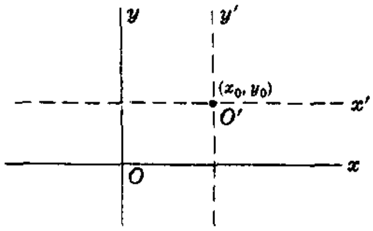

Fig. 8-6

TRANSFORMATION OF COORDINATES INVOLVING PURE ROTATION

8.12 $\begin{cases} x = x' \cos\alpha - y' \sin\alpha \\ y = x' \sin\alpha + y' \cos\alpha \end{cases}$ or $\begin{cases} x' = x \cos\alpha + y \sin\alpha \\ y' = y \cos\alpha - x \sin\alpha \end{cases}$

where the origins of the old $[xy]$ and new $[x'y']$ coordinate systems are the same but the x' axis makes an angle α with the positive x axis.

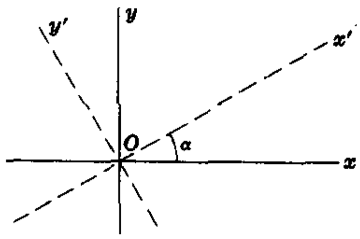

Fig. 8-7

TRANSFORMATION OF COORDINATES INVOLVING TRANSLATION AND ROTATION

8.13 $\begin{cases} x = x' \cos\alpha - y' \sin\alpha + x_0 \\ y = x' \sin\alpha + y' \cos\alpha + y_0 \end{cases}$

or $\begin{cases} x' = (x - x_0) \cos\alpha + (y - y_0) \sin\alpha \\ y' = (y - y_0) \cos\alpha - (x - x_0) \sin\alpha \end{cases}$

where the new origin O' of $x'y'$ coordinate system has coordinates (x_0, y_0) relative to the old xy coordinate system and the x' axis makes an angle α with the positive x axis.

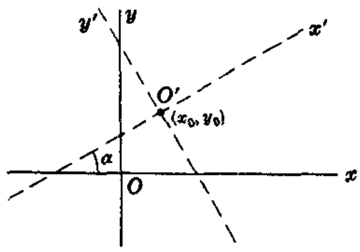

Fig. 8-8

POLAR COORDINATES (r, θ)

A point P can be located by rectangular coordinates (x, y) or polar coordinates (r, θ). The transformation between these coordinates is as follows:

8.14 $\begin{cases} x = r \cos\theta \\ y = r \sin\theta \end{cases}$ or $\begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1}(y/x) \end{cases}$

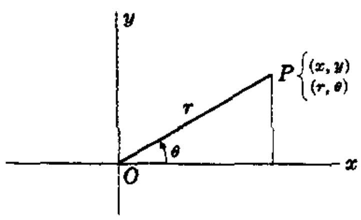

Fig. 8-9

EQUATION OF CIRCLE OF RADIUS R, CENTER AT (x_0, y_0)

8.15 $$(x - x_0)^2 + (y - y_0)^2 = R^2$$

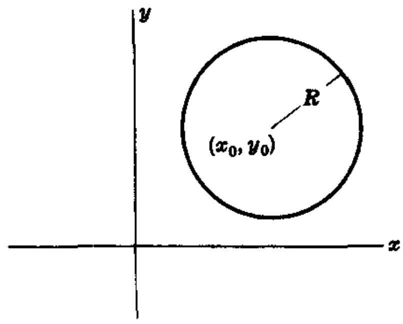

Fig. 8-10

EQUATION OF CIRCLE OF RADIUS R PASSING THROUGH ORIGIN

8.16 $$r = 2R\cos(\theta - \alpha)$$

where (r, θ) are polar coordinates of any point on the circle and (R, α) are polar coordinates of the center of the circle.

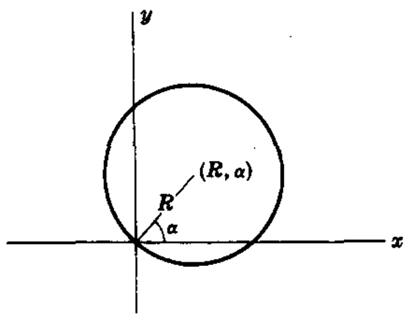

Fig. 8-11

CONICS (ELLIPSE, PARABOLA OR HYPERBOLA)

If a point P moves so that its distance from a fixed point [called the *focus*] divided by its distance from a fixed line [called the *directrix*] is a constant ϵ [called the *eccentricity*], then the curve described by P is called a *conic* [so-called because such curves can be obtained by intersecting a plane and a cone at different angles].

If the focus is chosen at origin O the equation of a conic in polar coordinates (r, θ) is, if $OQ = p$ and $LM = D$, [see Fig. 8-12]

8.17 $$r = \frac{p}{1 - \epsilon\cos\theta} = \frac{\epsilon D}{1 - \epsilon\cos\theta}$$

The conic is

(i) an ellipse if $\epsilon < 1$

(ii) a parabola if $\epsilon = 1$

(iii) a hyperbola if $\epsilon > 1$.

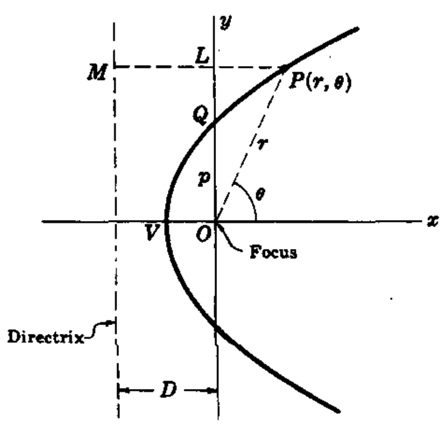

Fig. 8-12

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ELLIPSE WITH CENTER $C(x_0, y_0)$ AND MAJOR AXIS PARALLEL TO x AXIS

8.18 Length of major axis $A'A = 2a$

8.19 Length of minor axis $B'B = 2b$

8.20 Distance from center C to focus F or F' is

$$c = \sqrt{a^2 - b^2}$$

8.21 Eccentricity $= \epsilon = \dfrac{c}{a} = \dfrac{\sqrt{a^2 - b^2}}{a}$

8.22 Equation in rectangular coordinates:

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} = 1$$

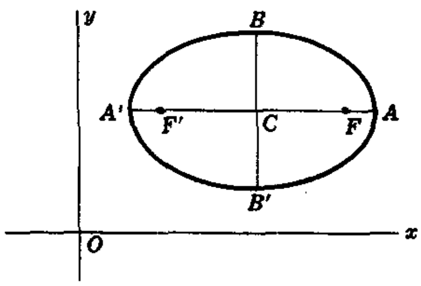

Fig. 8-13

8.23 Equation in polar coordinates if C is at O: $r^2 = \dfrac{a^2 b^2}{a^2 \sin^2 \theta + b^2 \cos^2 \theta}$

8.24 Equation in polar coordinates if C is on x axis and F' is at O: $r = \dfrac{a(1 - \epsilon^2)}{1 - \epsilon \cos \theta}$

8.25 If P is any point on the ellipse, $PF + PF' = 2a$

If the major axis is parallel to the y axis, interchange x and y in the above or replace θ by $\frac{1}{2}\pi - \theta$ [or $90° - \theta$].

PARABOLA WITH AXIS PARALLEL TO x AXIS

If vertex is at $A(x_0, y_0)$ and the distance from A to focus F is $a > 0$, the equation of the parabola is

8.26 $(y - y_0)^2 = 4a(x - x_0)$ if parabola opens to right [Fig. 8-14]

8.27 $(y - y_0)^2 = -4a(x - x_0)$ if parabola opens to left [Fig. 8-15]

If focus is at the origin [Fig. 8-16] the equation in polar coordinates is

8.28
$$r = \frac{2a}{1 - \cos \theta}$$

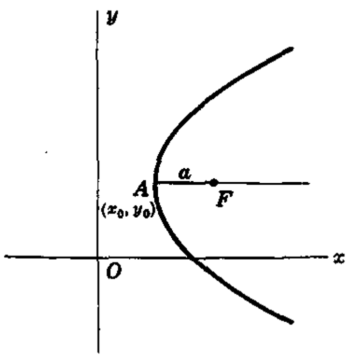

Fig. 8-14

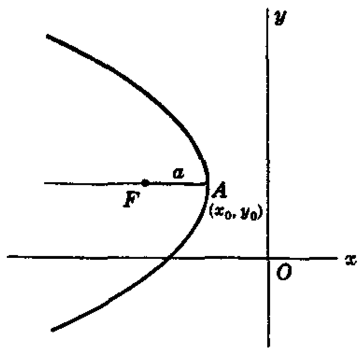

Fig. 8-15

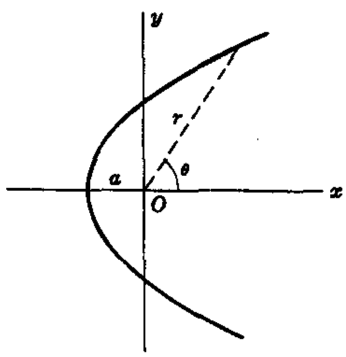

Fig. 8-16

In case the axis is parallel to the y axis, interchange x and y or replace θ by $\frac{1}{2}\pi - \theta$ [or $90° - \theta$].

HYPERBOLA WITH CENTER $C(x_0, y_0)$ AND MAJOR AXIS PARALLEL TO x AXIS

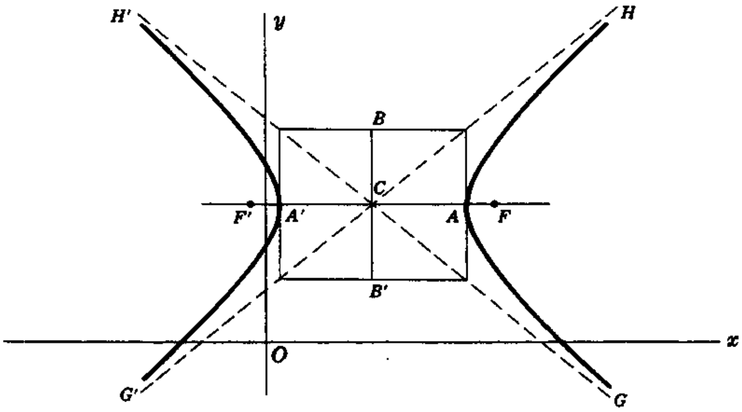

Fig. 8-17

8.29 Length of major axis $A'A = 2a$

8.30 Length of minor axis $B'B = 2b$

8.31 Distance from center C to focus F or $F' = c = \sqrt{a^2 + b^2}$

8.32 Eccentricity $\epsilon = \dfrac{c}{a} = \dfrac{\sqrt{a^2+b^2}}{a}$

8.33 Equation in rectangular coordinates: $\dfrac{(x-x_0)^2}{a^2} - \dfrac{(y-y_0)^2}{b^2} = 1$

8.34 Slopes of asymptotes $G'H$ and $GH' = \pm\dfrac{b}{a}$

8.35 Equation in polar coordinates if C is at O: $r^2 = \dfrac{a^2b^2}{b^2\cos^2\theta - a^2\sin^2\theta}$

8.36 Equation in polar coordinates if C is on x axis and F' is at O: $r = \dfrac{a(\epsilon^2 - 1)}{1 - \epsilon\cos\theta}$

8.37 If P is any point on the hyperbola, $PF - PF' = \pm 2a$ [depending on branch]

If the major axis is parallel to the y axis, interchange x and y in the above or replace θ by $\frac{1}{2}\pi - \theta$ [or $90° - \theta$].

9 SPECIAL PLANE CURVES

LEMNISCATE

9.1 Equation in polar coordinates:

$$r^2 = a^2 \cos 2\theta$$

9.2 Equation in rectangular coordinates:

$$(x^2 + y^2)^2 = a^2(x^2 - y^2)$$

9.3 Angle between AB' or $A'B$ and x axis = 45°

9.4 Area of one loop = a^2

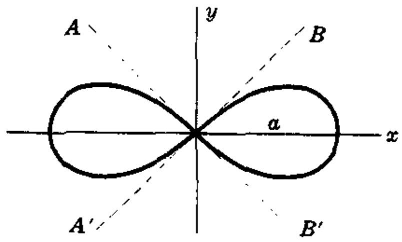

Fig. 9-1

CYCLOID

9.5 Equations in parametric form:

$$\begin{cases} x = a(\phi - \sin \phi) \\ y = a(1 - \cos \phi) \end{cases}$$

9.6 Area of one arch = $3\pi a^2$

9.7 Arc length of one arch = $8a$

This is a curve described by a point P on a circle of radius a rolling along x axis.

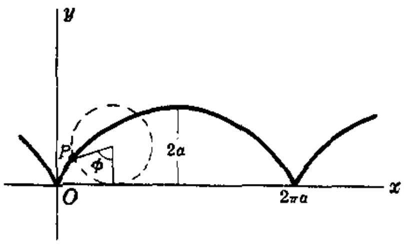

Fig. 9-2

HYPOCYCLOID WITH FOUR CUSPS

9.8 Equation in rectangular coordinates:

$$x^{2/3} + y^{2/3} = a^{2/3}$$

9.9 Equations in parametric form:

$$\begin{cases} x = a \cos^3 \theta \\ y = a \sin^3 \theta \end{cases}$$

9.10 Area bounded by curve = $\frac{3}{8}\pi a^2$

9.11 Arc length of entire curve = $6a$

This is a curve described by a point P on a circle of radius $a/4$ as it rolls on the inside of a circle of radius a.

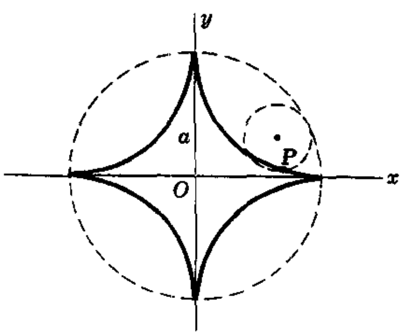

Fig. 9-3

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CARDIOID

9.12 Equation: $r = 2a(1 + \cos\theta)$

9.13 Area bounded by curve $= 6\pi a^2$

9.14 Arc length of curve $= 16a$

This is the curve described by a point P of a circle of radius a as it rolls on the outside of a fixed circle of radius a. The curve is also a special case of the limacon of Pascal [see 9.32].

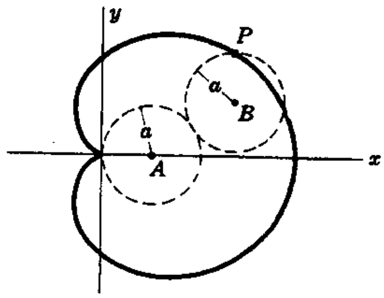

Fig. 9-4

CATENARY

9.15 Equation: $y = \dfrac{a}{2}(e^{x/a} + e^{-x/a}) = a\cosh\dfrac{x}{a}$

This is the curve in which a heavy uniform chain would hang if suspended vertically from fixed points A and B.

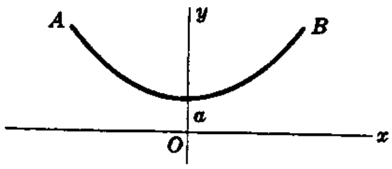

Fig. 9-5

THREE-LEAVED ROSE

9.16 Equation: $r = a\cos 3\theta$

The equation $r = a\sin 3\theta$ is a similar curve obtained by rotating the curve of Fig. 9-6 counterclockwise through 30° or $\pi/6$ radians.

In general $r = a\cos n\theta$ or $r = a\sin n\theta$ has n leaves if n is odd.

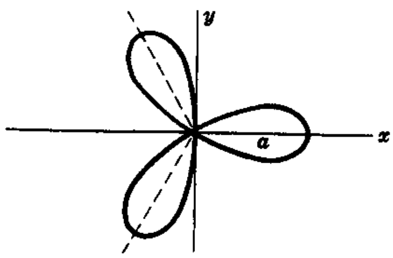

Fig. 9-6

FOUR-LEAVED ROSE

9.17 Equation: $r = a\cos 2\theta$

The equation $r = a\sin 2\theta$ is a similar curve obtained by rotating the curve of Fig. 9-7 counterclockwise through 45° or $\pi/4$ radians.

In general $r = a\cos n\theta$ or $r = a\ \sin n\theta$ has $2n$ leaves if n is even.

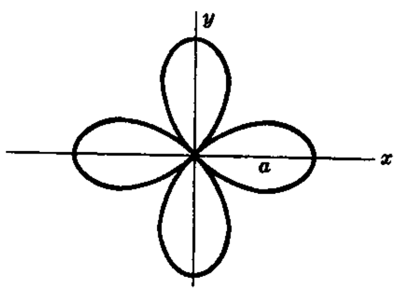

Fig. 9-7

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EPICYCLOID

9.18 Parametric equations:

$$\begin{cases} x = (a+b)\cos\theta - b\cos\left(\dfrac{a+b}{b}\right)\theta \\ y = (a+b)\sin\theta - b\sin\left(\dfrac{a+b}{b}\right)\theta \end{cases}$$

This is the curve described by a point P on a circle of radius b as it rolls on the outside of a circle of radius a.

The cardioid [Fig. 9-4] is a special case of an epicycloid.

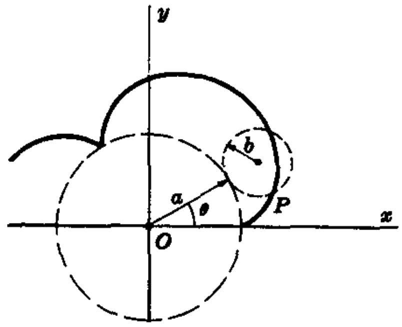

Fig. 9-8

GENERAL HYPOCYCLOID

9.19 Parametric equations:

$$\begin{cases} x = (a-b)\cos\phi + b\cos\left(\dfrac{a-b}{b}\right)\phi \\ y = (a-b)\sin\phi - b\sin\left(\dfrac{a-b}{b}\right)\phi \end{cases}$$

This is the curve described by a point P on a circle of radius b as it rolls on the inside of a circle of radius a.

If $b = a/4$, the curve is that of Fig. 9-3.

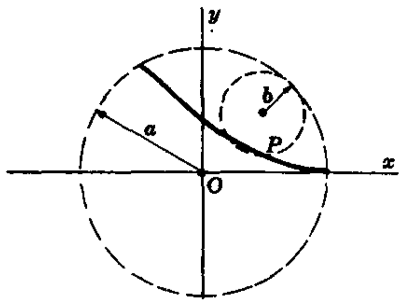

Fig. 9-9

TROCHOID

9.20 Parametric equations: $\begin{cases} x = a\phi - b\sin\phi \\ y = a - b\cos\phi \end{cases}$

This is the curve described by a point P at distance b from the center of a circle of radius a as the circle rolls on the x axis.

If $b < a$, the curve is as shown in Fig. 9-10 and is called a *curtate cycloid*.

If $b > a$, the curve is as shown in Fig. 9-11 and is called a *prolate cycloid*.

If $b = a$, the curve is the cycloid of Fig. 9-2.

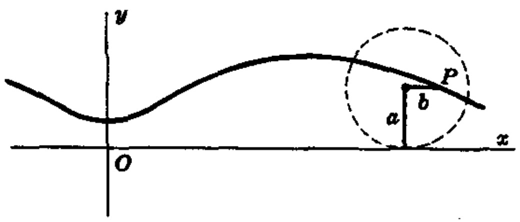

Fig. 9-10

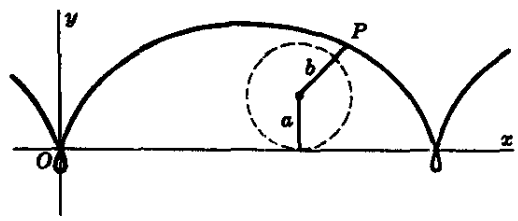

Fig. 9-11

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TRACTRIX

9.21 Parametric equations: $\begin{cases} x = a(\ln \cot \frac{1}{2}\phi - \cos \phi) \\ y = a \sin \phi \end{cases}$

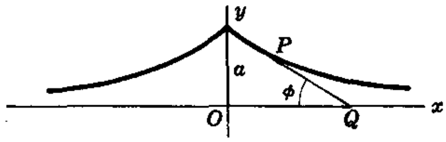

Fig. 9-12

This is the curve described by endpoint P of a taut string PQ of length a as the other end Q is moved along the x axis.

WITCH OF AGNESI

9.22 Equation in rectangular coordinates: $y = \dfrac{8a^3}{x^2 + 4a^2}$

9.23 Parametric equations: $\begin{cases} x = 2a \cot \theta \\ y = a(1 - \cos 2\theta) \end{cases}$

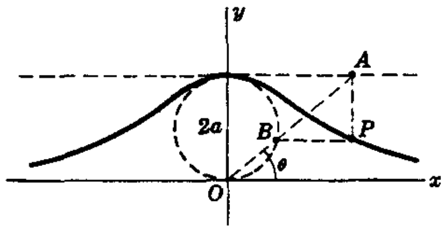

Fig. 9-13

In Fig. 9-13 the variable line OA intersects $y = 2a$ and the circle of radius a with center $(0, a)$ at A and B respectively. Any point P on the "witch" is located by constructing lines parallel to the x and y axes through B and A respectively and determining the point P of intersection.

FOLIUM OF DESCARTES

9.24 Equation in rectangular coordinates:

$$x^3 + y^3 = 3axy$$

9.25 Parametric equations:

$$\begin{cases} x = \dfrac{3at}{1+t^3} \\ y = \dfrac{3at^2}{1+t^3} \end{cases}$$

9.26 Area of loop $= \dfrac{3}{2}a^2$

9.27 Equation of asymptote: $x + y + a = 0$

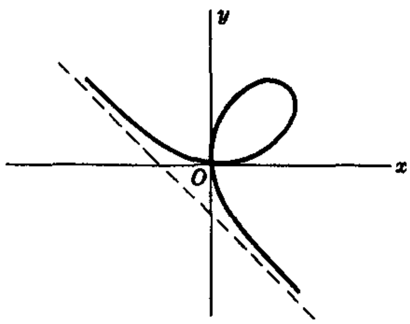

Fig. 9-14

INVOLUTE OF A CIRCLE

9.28 Parametric equations:

$$\begin{cases} x = a(\cos \phi + \phi \sin \phi) \\ y = a(\sin \phi - \phi \cos \phi) \end{cases}$$

This is the curve described by the endpoint P of a string as it unwinds from a circle of radius a while held taut.

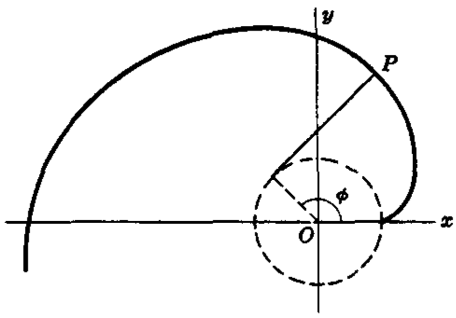

Fig. 9-15

EVOLUTE OF AN ELLIPSE

9.29 Equation in rectangular coordinates:

$$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$$

9.30 Parametric equations:

$$\begin{cases} ax = (a^2 - b^2)\cos^3\theta \\ by = (a^2 - b^2)\sin^3\theta \end{cases}$$

This curve is the envelope of the normals to the ellipse $x^2/a^2 + y^2/b^2 = 1$ shown dashed in Fig. 9-16.

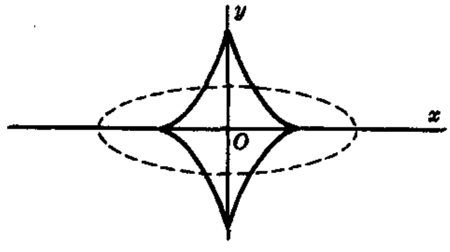

Fig. 9-16

OVALS OF CASSINI

9.31 Polar equation: $r^4 + a^4 - 2a^2r^2\cos 2\theta = b^4$

This is the curve described by a point P such that the product of its distance from two fixed points [distance $2a$ apart] is a constant b^2.

The curve is as in Fig. 9-17 or Fig. 9-18 according as $b < a$ or $b > a$ respectively.

If $b = a$, the curve is a *lemniscate* [Fig. 9-1].

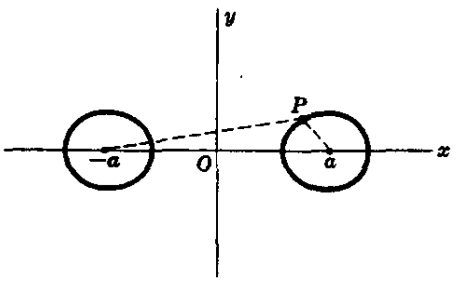

Fig. 9-17

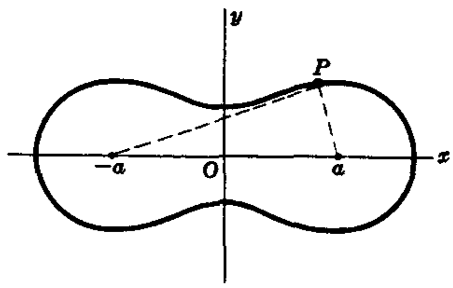

Fig. 9-18

LIMACON OF PASCAL

9.32 Polar equation: $r = b + a\cos\theta$

Let OQ be a line joining origin O to any point Q on a circle of diameter a passing through O. Then the curve is the locus of all points P such that $PQ = b$.

The curve is as in Fig. 9-19 or Fig. 9-20 according as $2a > b > a$ or $b < a$ respectively. If $b = a$, the curve is a *cardioid* [Fig. 9-4]. If $b \geqq 2a$, the curve is convex.

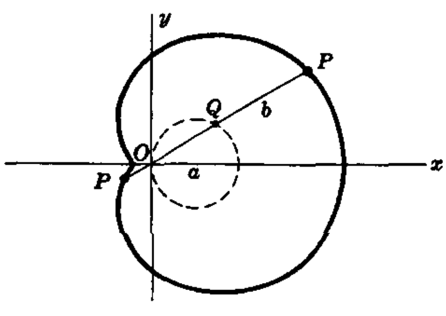

Fig. 9-19

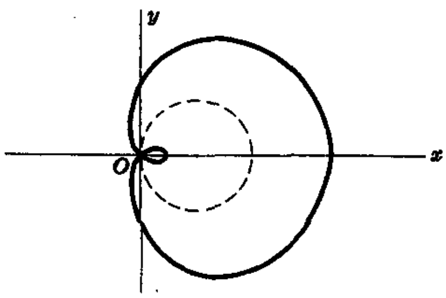

Fig. 9-20

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CISSOID OF DIOCLES

9.33 Equation in rectangular coordinates:

$$y^2 = \frac{x^3}{2a - x}$$

9.34 Parametric equations:

$$\begin{cases} x = 2a\sin^2\theta \\ y = \dfrac{2a\sin^3\theta}{\cos\theta} \end{cases}$$

This is the curve described by a point P such that the distance OP = distance RS. It is used in the problem of *duplication of a cube*, i.e. finding the side of a cube which has twice the volume of a given cube.

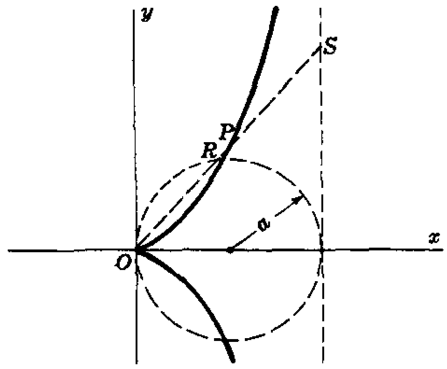

Fig. 9-21

SPIRAL OF ARCHIMEDES

9.35 Polar equation: $r = a\theta$

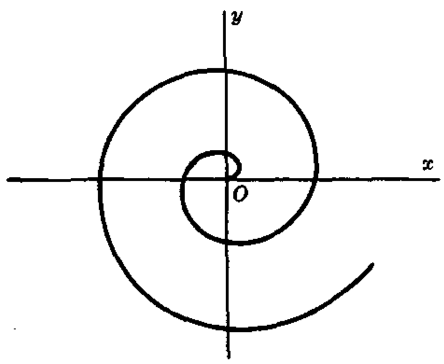

Fig. 9-22

10 FORMULAS from SOLID ANALYTIC GEOMETRY

DISTANCE d BETWEEN TWO POINTS $P_1(x_1, y_1, z_1)$ AND $P_2(x_2, y_2, z_2)$

10.1 $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

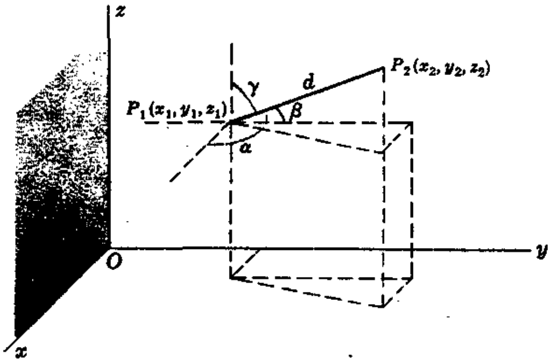

Fig. 10-1

DIRECTION COSINES OF LINE JOINING POINTS $P_1(x_1, y_1, z_1)$ AND $P_2(x_2, y_2, z_2)$

10.2 $$l = \cos\alpha = \frac{x_2 - x_1}{d}, \quad m = \cos\beta = \frac{y_2 - y_1}{d}, \quad n = \cos\gamma = \frac{z_2 - z_1}{d}$$

where α, β, γ are the angles which line P_1P_2 makes with the positive x, y, z axes respectively and d is given by 10.1 [see Fig. 10-1].

RELATIONSHIP BETWEEN DIRECTION COSINES

10.3 $$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \quad \text{or} \quad l^2 + m^2 + n^2 = 1$$

DIRECTION NUMBERS

Numbers L, M, N which are proportional to the direction cosines l, m, n are called *direction numbers*. The relationship between them is given by

10.4 $$l = \frac{L}{\sqrt{L^2 + M^2 + N^2}}, \quad m = \frac{M}{\sqrt{L^2 + M^2 + N^2}}, \quad n = \frac{N}{\sqrt{L^2 + M^2 + N^2}}$$

EQUATIONS OF LINE JOINING $P_1(x_1, y_1, z_1)$ AND $P_2(x_2, y_2, z_2)$ IN STANDARD FORM

10.5 $$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} \quad \text{or} \quad \frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$$

These are also valid if l, m, n are replaced by L, M, N respectively.

EQUATIONS OF LINE JOINING $P_1(x_1, y_1, z_1)$ AND $P_2(x_2, y_2, z_2)$ IN PARAMETRIC FORM

10.6 $$x = x_1 + lt, \quad y = y_1 + mt, \quad z = z_1 + nt$$

These are also valid if l, m, n are replaced by L, M, N respectively.

ANGLE ϕ BETWEEN TWO LINES WITH DIRECTION COSINES l_1, m_1, n_1 AND l_2, m_2, n_2

10.7 $$\cos\phi = l_1 l_2 + m_1 m_2 + n_1 n_2$$

GENERAL EQUATION OF A PLANE

10.8 $$Ax + By + Cz + D = 0$$ [A, B, C, D are constants]

EQUATION OF PLANE PASSING THROUGH POINTS (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)

10.9 $$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

or

10.10 $$\begin{vmatrix} y_2-y_1 & z_2-z_1 \\ y_3-y_1 & z_3-z_1 \end{vmatrix}(x-x_1) + \begin{vmatrix} z_2-z_1 & x_2-x_1 \\ z_3-z_1 & x_3-x_1 \end{vmatrix}(y-y_1) + \begin{vmatrix} x_2-x_1 & y_2-y_1 \\ x_3-x_1 & y_3-y_1 \end{vmatrix}(z-z_1) = 0$$

EQUATION OF PLANE IN INTERCEPT FORM

10.11 $$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

where a, b, c are the intercepts on the x, y, z axes respectively.

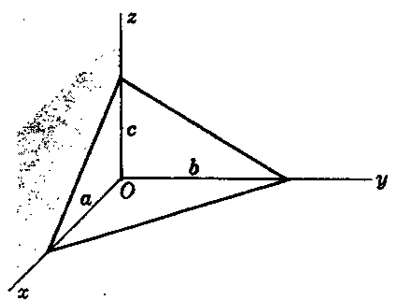

Fig. 10-2

Dr. Gilberto Paredes

EQUATIONS OF LINE THROUGH (x_0, y_0, z_0) AND PERPENDICULAR TO PLANE $Ax + By + Cz + D = 0$

10.12 $$\frac{x - x_0}{A} = \frac{y - y_0}{B} = \frac{z - z_0}{C} \quad \text{or} \quad x = x_0 + At, \quad y = y_0 + Bt, \quad z = z_0 + Ct$$

Note that the direction numbers for a line perpendicular to the plane $Ax + By + Cz + D = 0$ are A, B, C.

DISTANCE FROM POINT (x_0, y_0, z_0) TO PLANE $Ax + By + Cz + D = 0$

10.13 $$\frac{Ax_0 + By_0 + Cz_0 + D}{\pm\sqrt{A^2 + B^2 + C^2}}$$

where the sign is chosen so that the distance is nonnegative.

NORMAL FORM FOR EQUATION OF PLANE

10.14 $$x \cos\alpha + y \cos\beta + z \cos\gamma = p$$

where p = perpendicular distance from O to plane at P and α, β, γ are angles between OP and positive x, y, z axes.

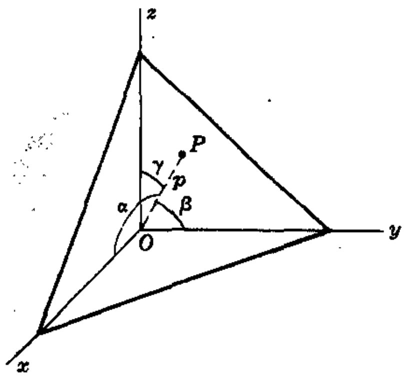

Fig. 10-3

TRANSFORMATION OF COORDINATES INVOLVING PURE TRANSLATION

10.15 $$\begin{cases} x = x' + x_0 \\ y = y' + y_0 \\ z = z' + z_0 \end{cases} \quad \text{or} \quad \begin{cases} x' = x - x_0 \\ y' = y - y_0 \\ z' = z - z_0 \end{cases}$$

where (x, y, z) are old coordinates [i.e. coordinates relative to xyz system], (x', y', z') are new coordinates [relative to $x'y'z'$ system] and (x_0, y_0, z_0) are the coordinates of the new origin O' relative to the old xyz coordinate system.

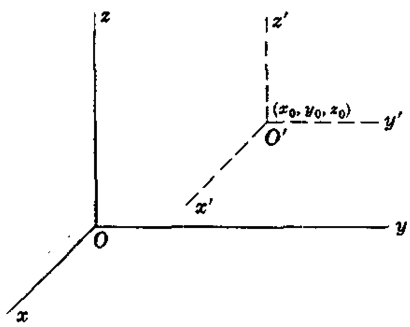

Fig. 10-4

TRANSFORMATION OF COORDINATES INVOLVING PURE ROTATION

10.16
$$\begin{cases} x = l_1x' + l_2y' + l_3z' \\ y = m_1x' + m_2y' + m_3z' \\ z = n_1x' + n_2y' + n_3z' \end{cases}$$

or
$$\begin{cases} x' = l_1x + m_1y + n_1z \\ y' = l_2x + m_2y + n_2z \\ z' = l_3x + m_3y + n_3z \end{cases}$$

where the origins of the xyz and $x'y'z'$ systems are the same and l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 are the direction cosines of the x', y', z' axes relative to the x, y, z axes respectively.

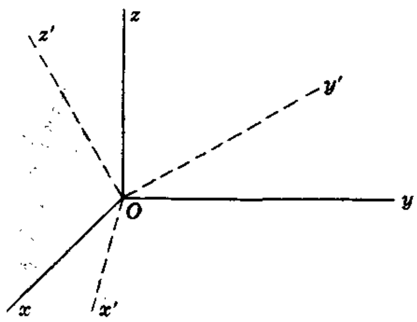

Fig. 10-5

TRANSFORMATION OF COORDINATES INVOLVING TRANSLATION AND ROTATION

10.17
$$\begin{cases} x = l_1x' + l_2y' + l_3z' + x_0 \\ y = m_1x' + m_2y' + m_3z' + y_0 \\ z = n_1x' + n_2y' + n_3z' + z_0 \end{cases}$$

or
$$\begin{cases} x' = l_1(x - x_0) + m_1(y - y_0) + n_1(z - z_0) \\ y' = l_2(x - x_0) + m_2(y - y_0) + n_2(z - z_0) \\ z' = l_3(x - x_0) + m_3(y - y_0) + n_3(z - z_0) \end{cases}$$

where the origin O' of the $x'y'z'$ system has coordinates (x_0, y_0, z_0) relative to the xyz system and

$$l_1, m_1, n_1; \quad l_2, m_2, n_2; \quad l_3, m_3, n_3$$

are the direction cosines of the x', y', z' axes relative to the x, y, z axes respectively.

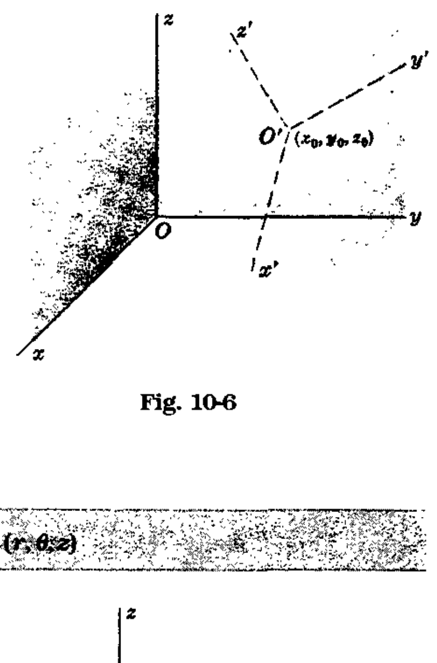

Fig. 10-6

CYLINDRICAL COORDINATES (r, θ, z)

A point P can be located by cylindrical coordinates (r, θ, z) [see Fig. 10-7] as well as rectangular coordinates (x, y, z).

The transformation between these coordinates is

10.18
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \quad \text{or} \quad \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1}(y/x) \\ z = z \end{cases}$$

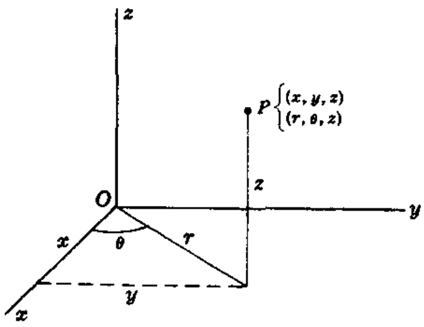

Fig. 10-7

SPHERICAL COORDINATES (r, θ, ϕ)

A point P can be located by spherical coordinates (r, θ, ϕ) [see Fig. 10-8] as well as rectangular coordinates (x, y, z).

The transformation between those coordinates is

10.19
$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$

or
$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \phi = \tan^{-1}(y/x) \\ \theta = \cos^{-1}(z/\sqrt{x^2+y^2+z^2}) \end{cases}$$

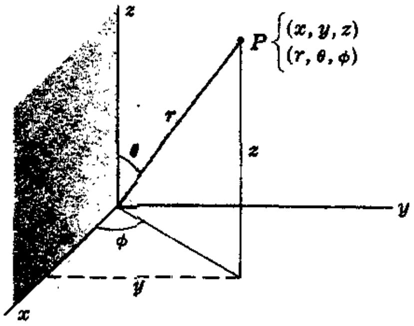

Fig. 10-8

EQUATION OF SPHERE IN RECTANGULAR COORDINATES

10.20
$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2 = R^2$$

where the sphere has center (x_0, y_0, z_0) and radius R.

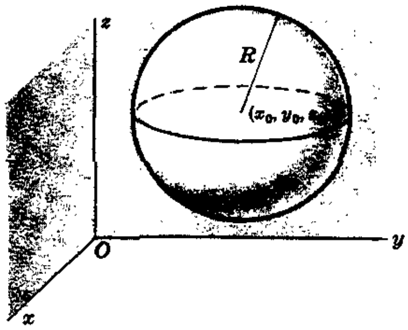

Fig. 10-9

EQUATION OF SPHERE IN CYLINDRICAL COORDINATES

10.21
$$r^2 - 2r_0r\cos(\theta-\theta_0) + r_0^2 + (z-z_0)^2 = R^2$$

where the sphere has center (r_0, θ_0, z_0) in cylindrical coordinates and radius R.

If the center is at the origin the equation is

10.22
$$r^2 + z^2 = R^2$$

EQUATION OF SPHERE IN SPHERICAL COORDINATES

10.23
$$r^2 + r_0^2 - 2r_0r\sin\theta\sin\theta_0\cos(\phi-\phi_0) = R^2$$

where the sphere has center (r_0, θ_0, ϕ_0) in spherical coordinates and radius R.

If the center is at the origin the equation is

10.24
$$r = R$$

EQUATION OF ELLIPSOID WITH CENTER (x_0, y_0, z_0) AND SEMI-AXES a, b, c

10.25 $$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} + \frac{(z-z_0)^2}{c^2} = 1$$

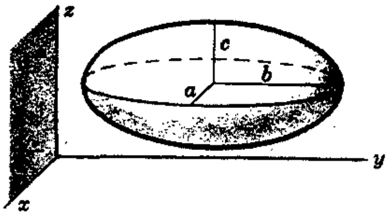

Fig. 10-10

ELLIPTIC CYLINDER WITH AXIS AS z AXIS

10.26 $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where a, b are semi-axes of elliptic cross section.

If $b = a$ it becomes a circular cylinder of radius a.

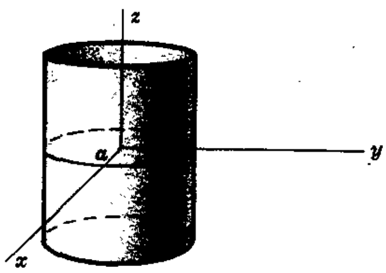

Fig. 10-11

ELLIPTIC CONE WITH AXIS AS z AXIS

10.27 $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

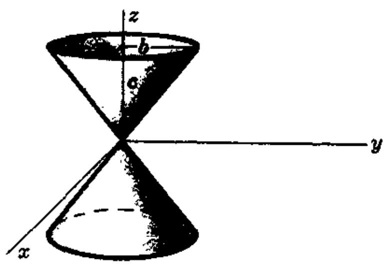

Fig. 10-12

HYPERBOLOID OF ONE SHEET

10.28 $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

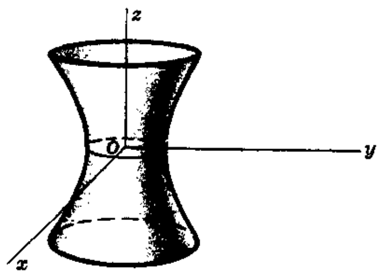

Fig. 10-13

HYPERBOLOID OF TWO SHEETS

10.29 $$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Note orientation of axes in Fig. 10-14.

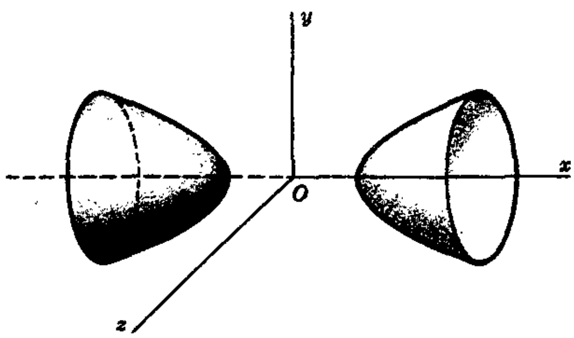

Fig. 10-14

ELLIPTIC PARABOLOID

10.30 $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$

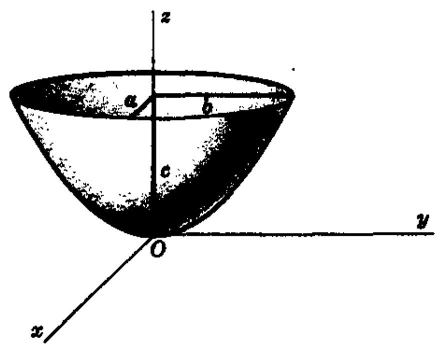

Fig. 10-15

HYPERBOLIC PARABOLOID

10.31 $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$$

Note orientation of axes in Fig. 10-16.

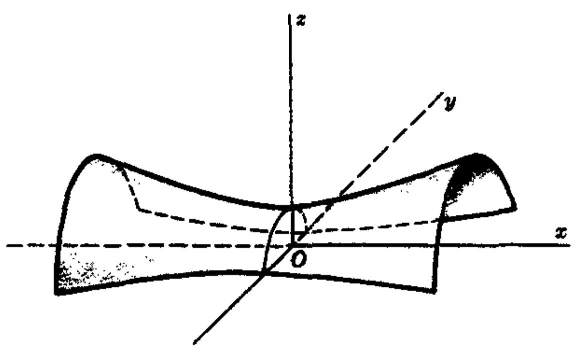

Fig. 10-16

11 SPECIAL MOMENTS OF INERTIA

The table below shows the moments of inertia of various rigid bodies of mass M. In all cases it is assumed the body has uniform [i.e. constant] density.

TYPE OF RIGID BODY	MOMENT OF INERTIA
11.1 Thin rod of length a	
(a) about axis perpendicular to the rod through the center of mass,	$\frac{1}{12}Ma^2$
(b) about axis perpendicular to the rod through one end.	$\frac{1}{3}Ma^2$
11.2 Rectangular parallelepiped with sides a, b, c	
(a) about axis parallel to c and through center of face ab,	$\frac{1}{12}M(a^2+b^2)$
(b) about axis through center of face bc and parallel to c.	$\frac{1}{12}M(4a^2+b^2)$
11.3 Thin rectangular plate with sides a, b	
(a) about axis perpendicular to the plate through center,	$\frac{1}{12}M(a^2+b^2)$
(b) about axis parallel to side b through center.	$\frac{1}{12}Ma^2$
11.4 Circular cylinder of radius a and height h	
(a) about axis of cylinder,	$\frac{1}{2}Ma^2$
(b) about axis through center of mass and perpendicular to cylindrical axis,	$\frac{1}{12}M(h^2+3a^2)$
(c) about axis coinciding with diameter at one end.	$\frac{1}{12}M(4h^2+3a^2)$
11.5 Hollow circular cylinder of outer radius a, inner radius b and height h	
(a) about axis of cylinder,	$\frac{1}{2}M(a^2+b^2)$
(b) about axis through center of mass and perpendicular to cylindrical axis,	$\frac{1}{12}M(3a^2+3b^2+h^2)$
(c) about axis coinciding with diameter at one end.	$\frac{1}{12}M(3a^2+3b^2+4h^2)$
11.6 Circular plate of radius a	
(a) about axis perpendicular to plate through center,	$\frac{1}{2}Ma^2$
(b) about axis coinciding with a diameter.	$\frac{1}{4}Ma^2$

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11.7 Hollow circular plate or ring with outer radius a and inner radius b	
(a) about axis perpendicular to plane of plate through center,	$\frac{1}{2}M(a^2+b^2)$
(b) about axis coinciding with a diameter.	$\frac{1}{4}M(a^2+b^2)$
11.8 Thin circular ring of radius a	
(a) about axis perpendicular to plane of ring through center,	Ma^2
(b) about axis coinciding with diameter.	$\frac{1}{2}Ma^2$
11.9 Sphere of radius a	
(a) about axis coinciding with a diameter,	$\frac{2}{5}Ma^2$
(b) about axis tangent to the surface.	$\frac{7}{5}Ma^2$
11.10 Hollow sphere of outer radius a and inner radius b	
(a) about axis coinciding with a diameter,	$\frac{2}{5}M(a^5-b^5)/(a^3-b^3)$
(b) about axis tangent to the surface.	$\frac{2}{5}M(a^5-b^5)/(a^3-b^3)+Ma^2$
11.11 Hollow spherical shell of radius a	
(a) about axis coinciding with a diameter,	$\frac{2}{3}Ma^2$
(b) about axis tangent to the surface.	$\frac{5}{3}Ma^2$
11.12 Ellipsoid with semi-axes a, b, c	
(a) about axis coinciding with semi-axis c,	$\frac{1}{5}M(a^2+b^2)$
(b) about axis tangent to surface, parallel to semi-axis c and at distance a from center.	$\frac{1}{5}M(6a^2+b^2)$
11.13 Circular cone of radius a and height h	
(a) about axis of cone,	$\frac{3}{10}Ma^2$
(b) about axis through vertex and perpendicular to axis,	$\frac{3}{20}M(a^2+4h^2)$
(c) about axis through center of mass and perpendicular to axis.	$\frac{3}{80}M(4a^2+h^2)$
11.14 Torus with outer radius a and inner radius b	
(a) about axis through center of mass and perpendicular to plane of torus,	$\frac{1}{4}M(7a^2-6ab+3b^2)$
(b) about axis through center of mass and in the plane of the torus.	$\frac{1}{4}M(9a^2-10ab+5b^2)$