Section IV: Calculus

15 DERIVATIVES

DEFINITION OF A DERIVATIVE

Suppose $y = f(x)$. The derivative of y or $f(x)$ is defined as

15.1 $$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

where $h = \Delta x$. The derivative is also denoted by y', df/dx or $f'(x)$. The process of taking a derivative is called *differentiation*.

[illegible]

In the following, u, v, w are functions of x; a, b, c, n are constants [restricted if indicated]; $e = 2.71828\ldots$ is the natural base of logarithms; $\ln u$ is the natural logarithm of u [i.e. the logarithm to the base e] where it is assumed that $u > 0$ and all angles are in radians.

15.2 $$\frac{d}{dx}(c) = 0$$

15.3 $$\frac{d}{dx}(cx) = c$$

15.4 $$\frac{d}{dx}(cx^n) = ncx^{n-1}$$

15.5 $$\frac{d}{dx}(u \pm v \pm w \pm \cdots) = \frac{du}{dx} \pm \frac{dv}{dx} \pm \frac{dw}{dx} \pm \cdots$$

15.6 $$\frac{d}{dx}(cu) = c\frac{du}{dx}$$

15.7 $$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

15.8 $$\frac{d}{dx}(uvw) = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}$$

15.9 $$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v(du/dx) - u(dv/dx)}{v^2}$$

15.10 $$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

15.11 $$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \quad \text{(Chain rule)}$$

15.12 $$\frac{du}{dx} = \frac{1}{dx/du}$$

15.13 $$\frac{dy}{dx} = \frac{dy/du}{dx/du}$$

DERIVATIVES OF TRIGONOMETRIC AND INVERSE TRIGONOMETRIC FUNCTIONS

15.14 $\dfrac{d}{dx}\sin u = \cos u \dfrac{du}{dx}$

15.15 $\dfrac{d}{dx}\cos u = -\sin u \dfrac{du}{dx}$

15.16 $\dfrac{d}{dx}\tan u = \sec^2 u \dfrac{du}{dx}$

15.17 $\dfrac{d}{dx}\cot u = -\csc^2 u \dfrac{du}{dx}$

15.18 $\dfrac{d}{dx}\sec u = \sec u \tan u \dfrac{du}{dx}$

15.19 $\dfrac{d}{dx}\csc u = -\csc u \cot u \dfrac{du}{dx}$

15.20 $\dfrac{d}{dx}\sin^{-1} u = \dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx} \qquad \left[-\dfrac{\pi}{2} < \sin^{-1} u < \dfrac{\pi}{2}\right]$

15.21 $\dfrac{d}{dx}\cos^{-1} u = \dfrac{-1}{\sqrt{1-u^2}}\dfrac{du}{dx} \qquad [0 < \cos^{-1} u < \pi]$

15.22 $\dfrac{d}{dx}\tan^{-1} u = \dfrac{1}{1+u^2}\dfrac{du}{dx} \qquad \left[-\dfrac{\pi}{2} < \tan^{-1} u < \dfrac{\pi}{2}\right]$

15.23 $\dfrac{d}{dx}\cot^{-1} u = \dfrac{-1}{1+u^2}\dfrac{du}{dx} \qquad [0 < \cot^{-1} u < \pi]$

15.24 $\dfrac{d}{dx}\sec^{-1} u = \dfrac{1}{|u|\sqrt{u^2-1}}\dfrac{du}{dx} = \dfrac{\pm 1}{u\sqrt{u^2-1}}\dfrac{du}{dx} \qquad \begin{bmatrix} + \text{ if } 0 < \sec^{-1} u < \pi/2 \\ - \text{ if } \pi/2 < \sec^{-1} u < \pi \end{bmatrix}$

15.25 $\dfrac{d}{dx}\csc^{-1} u = \dfrac{-1}{|u|\sqrt{u^2-1}}\dfrac{du}{dx} = \dfrac{\mp 1}{u\sqrt{u^2-1}}\dfrac{du}{dx} \qquad \begin{bmatrix} - \text{ if } 0 < \csc^{-1} u < \pi/2 \\ + \text{ if } -\pi/2 < \csc^{-1} u < 0 \end{bmatrix}$

DERIVATIVES OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

15.26 $\dfrac{d}{dx}\log_a u = \dfrac{\log_a e}{u}\dfrac{du}{dx} \qquad a \neq 0, 1$

15.27 $\dfrac{d}{dx}\ln u = \dfrac{d}{dx}\log_e u = \dfrac{1}{u}\dfrac{du}{dx}$

15.28 $\dfrac{d}{dx}a^u = a^u \ln a \dfrac{du}{dx}$

15.29 $\dfrac{d}{dx}e^u = e^u \dfrac{du}{dx}$

15.30 $\dfrac{d}{dx}u^v = \dfrac{d}{dx}e^{v \ln u} = e^{v \ln u}\dfrac{d}{dx}[v \ln u] = vu^{v-1}\dfrac{du}{dx} + u^v \ln u \dfrac{dv}{dx}$

DERIVATIVES OF HYPERBOLIC AND INVERSE HYPERBOLIC FUNCTIONS

15.31 $\frac{d}{dx}\sinh u = \cosh u \frac{du}{dx}$

15.32 $\frac{d}{dx}\cosh u = \sinh u \frac{du}{dx}$

15.33 $\frac{d}{dx}\tanh u = \operatorname{sech}^2 u \frac{du}{dx}$

15.34 $\frac{d}{dx}\coth u = -\operatorname{csch}^2 u \frac{du}{dx}$

15.35 $\frac{d}{dx}\operatorname{sech} u = -\operatorname{sech} u \tanh u \frac{du}{dx}$

15.36 $\frac{d}{dx}\operatorname{csch} u = -\operatorname{csch} u \coth u \frac{du}{dx}$

15.37 $\frac{d}{dx}\sinh^{-1} u = \frac{1}{\sqrt{u^2+1}} \frac{du}{dx}$

15.38 $\frac{d}{dx}\cosh^{-1} u = \frac{\pm 1}{\sqrt{u^2-1}} \frac{du}{dx}$ $\left[\begin{array}{l} + \text{ if } \cosh^{-1} u > 0,\ u > 1 \\ - \text{ if } \cosh^{-1} u < 0,\ u > 1 \end{array}\right]$

15.39 $\frac{d}{dx}\tanh^{-1} u = \frac{1}{1-u^2} \frac{du}{dx}$ $[-1 < u < 1]$

15.40 $\frac{d}{dx}\coth^{-1} u = \frac{1}{1-u^2} \frac{du}{dx}$ $[u > 1 \text{ or } u < -1]$

15.41 $\frac{d}{dx}\operatorname{sech}^{-1} u = \frac{\mp 1}{u\sqrt{1-u^2}} \frac{du}{dx}$ $\left[\begin{array}{l} - \text{ if } \operatorname{sech}^{-1} u > 0,\ 0 < u < 1 \\ + \text{ if } \operatorname{sech}^{-1} u < 0,\ 0 < u < 1 \end{array}\right]$

15.42 $\frac{d}{dx}\operatorname{csch}^{-1} u = \frac{-1}{|u|\sqrt{1+u^2}} \frac{du}{dx} = \frac{\mp 1}{u\sqrt{1+u^2}} \frac{du}{dx}$ $[- \text{ if } u > 0, + \text{ if } u < 0]$

HIGHER DERIVATIVES

The second, third and higher derivatives are defined as follows.

15.43 Second derivative $= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = f''(x) = y''$

15.44 Third derivative $= \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} = f'''(x) = y'''$

15.45 nth derivative $= \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d^ny}{dx^n} = f^{(n)}(x) = y^{(n)}$

LEIBNIZ'S RULE FOR HIGHER DERIVATIVES OF PRODUCTS

Let D^p stand for the operator $\frac{d^p}{dx^p}$ so that $D^p u = \frac{d^p u}{dx^p}$ = the pth derivative of u. Then

15.46 $$D^n(uv) = uD^n v + \binom{n}{1}(Du)(D^{n-1}v) + \binom{n}{2}(D^2u)(D^{n-2}v) + \cdots + vD^n u$$

where $\binom{n}{1}, \binom{n}{2}, \ldots$ are the binomial coefficients [see 3.5].

As special cases we have

15.47 $$\frac{d^2}{dx^2}(uv) = u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2}$$

15.48 $$\frac{d^3}{dx^3}(uv) = u\frac{d^3v}{dx^3} + 3\frac{du}{dx}\frac{d^2v}{dx^2} + 3\frac{d^2u}{dx^2}\frac{dv}{dx} + v\frac{d^3u}{dx^3}$$

DIFFERENTIALS

Let $y = f(x)$ and $\Delta y = f(x + \Delta x) - f(x)$. Then

15.49 $$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x) + \epsilon = \frac{dy}{dx} + \epsilon$$

where $\epsilon \to 0$ as $\Delta x \to 0$. Thus

15.50 $$\Delta y = f'(x)\Delta x + \epsilon\Delta x$$

If we call $\Delta x = dx$ the differential of x, then we define the differential of y to be

15.51 $$dy = f'(x)\,dx$$

RULES FOR DIFFERENTIALS

The rules for differentials are exactly analogous to those for derivatives. As examples we observe that

15.52 $$d(u \pm v \pm w \pm \cdots) = du \pm dv \pm dw \pm \cdots$$

15.53 $$d(uv) = u\,dv + v\,du$$

15.54 $$d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$$

15.55 $$d(u^n) = nu^{n-1}\,du$$

15.56 $$d(\sin u) = \cos u\,du$$

15.57 $$d(\cos u) = -\sin u\,du$$

PARTIAL DERIVATIVES

Let $z = f(x, y)$ be a function of the two variables x and y. Then we define the *partial derivative* of z or $f(x, y)$ with respect to x, keeping y constant, to be

15.58 $$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

This partial derivative is also denoted by $\partial z/\partial x$, f_x, or z_x.

Similarly the partial derivative of $z = f(x, y)$ with respect to y, keeping x constant, is defined to be

15.59 $$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

This partial derivative is also denoted by $\partial z/\partial y$, f_y, or z_y.

Partial derivatives of higher order can be defined as follows:

15.60 $$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

15.61 $$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \quad \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

The results in 15.61 will be equal if the function and its partial derivatives are continuous; that is, in such cases, the order of differentiation makes no difference.

Extensions to functions of more than two variables are exactly analogous.

MULTIVARIABLE DIFFERENTIALS

The differential of $z = f(x, y)$ is defined as

15.62 $$dz = df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

where $dx = \Delta x$ and $dy = \Delta y$. Note that dz is a function of four variables, namely x, y, dx, dy, and is linear in the variables dx and dy.

Extensions to functions of more than two variables are exactly analogous.

Example: Let $z = x^2 + 5xy + 2y^3$. Then

$$z_x = 2x + 5y \quad \text{and} \quad z_y = 5x + 6y^2$$

and hence

$$dz = (2x + 5y)\,dx + (5x + 6y^2)\,dy$$

Suppose we want to find dz for $dx = 2$, $dy = 3$ and at the point P (4, 1), *i.e.* when $x = 4$ and $y = 1$. Substitution yields

$$dz = (8 + 5)2 + (20 + 6)3 = 26 + 78 = 104$$

16 INDEFINITE INTEGRALS

DEFINITION OF AN INDEFINITE INTEGRAL

If $\frac{dy}{dx} = f(x)$, then y is the function whose derivative is $f(x)$ and is called the *anti-derivative* of $f(x)$ or the *indefinite integral* of $f(x)$, denoted by $\int f(x)\,dx$. Similarly if $y = \int f(u)\,du$, then $\frac{dy}{du} = f(u)$. Since the derivative of a constant is zero, all indefinite integrals differ by an arbitrary constant.

For the definition of a definite integral, see 18.1. The process of finding an integral is called *integration*.

GENERAL RULES OF INTEGRATION

In the following, u, v, w are functions of x; a, b, p, q, n any constants, restricted if indicated; $e = 2.71828\ldots$ is the natural base of logarithms; $\ln u$ denotes the natural logarithm of u where it is assumed that $u > 0$ [in general, to extend formulas to cases where $u < 0$ as well, replace $\ln u$ by $\ln|u|$]; all angles are in radians; all constants of integration are omitted but implied.

16.1 $\int a\,dx = ax$

16.2 $\int af(x)\,dx = a\int f(x)\,dx$

16.3 $\int (u \pm v \pm w \pm \cdots)\,dx = \int u\,dx \pm \int v\,dx \pm \int w\,dx \pm \cdots$

16.4 $\int u\,dv = uv - \int v\,du$ [Integration by parts]

For generalized integration by parts, see 16.48.

16.5 $\int f(ax)\,dx = \frac{1}{a}\int f(u)\,du$

16.6 $\int F\{f(x)\}\,dx = \int F(u)\frac{dx}{du}\,du = \int \frac{F(u)}{f'(x)}\,du$ where $u = f(x)$

16.7 $\int u^n\,du = \frac{u^{n+1}}{n+1}, \quad n \neq -1$ [For $n = -1$, see 16.8]

16.8 $\int \frac{du}{u} = \ln u$ if $u > 0$ or $\ln(-u)$ if $u < 0$

$= \ln|u|$

16.9 $\int e^u\,du = e^u$

16.10 $\int a^u\,du = \int e^{u\ln a}\,du = \frac{e^{u\ln a}}{\ln a} = \frac{a^u}{\ln a}, \quad a > 0,\ a \neq 1$

16.11 $\int \sin u\,du = -\cos u$

16.12 $\int \cos u\,du = \sin u$

16.13 $\int \tan u\,du = \ln \sec u = -\ln \cos u$

16.14 $\int \cot u\,du = \ln \sin u$

16.15 $\int \sec u\,du = \ln(\sec u + \tan u) = \ln \tan\left(\frac{u}{2} + \frac{\pi}{4}\right)$

16.16 $\int \csc u\,du = \ln(\csc u - \cot u) = \ln \tan \frac{u}{2}$

16.17 $\int \sec^2 u\,du = \tan u$

16.18 $\int \csc^2 u\,du = -\cot u$

16.19 $\int \tan^2 u\,du = \tan u - u$

16.20 $\int \cot^2 u\,du = -\cot u - u$

16.21 $\int \sin^2 u\,du = \frac{u}{2} - \frac{\sin 2u}{4} = \frac{1}{2}(u - \sin u \cos u)$

16.22 $\int \cos^2 u\,du = \frac{u}{2} + \frac{\sin 2u}{4} = \frac{1}{2}(u + \sin u \cos u)$

16.23 $\int \sec u \tan u\,du = \sec u$

16.24 $\int \csc u \cot u\,du = -\csc u$

16.25 $\int \sinh u\,du = \cosh u$

16.26 $\int \cosh u\,du = \sinh u$

16.27 $\int \tanh u\,du = \ln \cosh u$

16.28 $\int \coth u\,du = \ln \sinh u$

16.29 $\int \operatorname{sech} u\,du = \sin^{-1}(\tanh u) \quad \text{or} \quad 2\tan^{-1} e^u$

16.30 $\int \operatorname{csch} u\,du = \ln \tanh \frac{u}{2} \quad \text{or} \quad -\coth^{-1} e^u$

16.31 $\int \operatorname{sech}^2 u\, du = \tanh u$

16.32 $\int \operatorname{csch}^2 u\, du = -\coth u$

16.33 $\int \tanh^2 u\, du = u - \tanh u$

16.34 $\int \coth^2 u\, du = u - \coth u$

16.35 $\int \sinh^2 u\, du = \frac{\sinh 2u}{4} - \frac{u}{2} = \frac{1}{2}(\sinh u \cosh u - u)$

16.36 $\int \cosh^2 u\, du = \frac{\sinh 2u}{4} + \frac{u}{2} = \frac{1}{2}(\sinh u \cosh u + u)$

16.37 $\int \operatorname{sech} u \tanh u\, du = -\operatorname{sech} u$

16.38 $\int \operatorname{csch} u \coth u\, du = -\operatorname{csch} u$

16.39 $\int \frac{du}{u^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{u}{a}$

16.40 $\int \frac{du}{u^2 - a^2} = \frac{1}{2a}\ln\left(\frac{u-a}{u+a}\right) = -\frac{1}{a}\coth^{-1}\frac{u}{a} \quad u^2 > a^2$

16.41 $\int \frac{du}{a^2 - u^2} = \frac{1}{2a}\ln\left(\frac{a+u}{a-u}\right) = \frac{1}{a}\tanh^{-1}\frac{u}{a} \quad u^2 < a^2$

16.42 $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a}$

16.43 $\int \frac{du}{\sqrt{u^2 + a^2}} = \ln(u + \sqrt{u^2 + a^2}) \quad \text{or} \quad \sinh^{-1}\frac{u}{a}$

16.44 $\int \frac{du}{\sqrt{u^2 - a^2}} = \ln(u + \sqrt{u^2 - a^2})$

16.45 $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right|$

16.46 $\int \frac{du}{u\sqrt{u^2 + a^2}} = -\frac{1}{a}\ln\left(\frac{a + \sqrt{u^2 + a^2}}{u}\right)$

16.47 $\int \frac{du}{u\sqrt{a^2 - u^2}} = -\frac{1}{a}\ln\left(\frac{a + \sqrt{a^2 - u^2}}{u}\right)$

16.48 $\int f^{(n)} g\, dx = f^{(n-1)}g - f^{(n-2)}g' + f^{(n-3)}g'' - \cdots (-1)^n \int f g^{(n)}\, dx$

This is called *generalized integration* by parts.

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IMPORTANT TRANSFORMATIONS

Often in practice an integral can be simplified by using an appropriate transformation or substitution together with Formula 16.6. The following list gives some transformations and their effects.

16.49 $\int F(ax+b)\,dx = \frac{1}{a}\int F(u)\,du$ where $u = ax+b$

16.50 $\int F(\sqrt{ax+b})\,dx = \frac{2}{a}\int u\,F(u)\,du$ where $u = \sqrt{ax+b}$

16.51 $\int F(\sqrt[n]{ax+b})\,dx = \frac{n}{a}\int u^{n-1}F(u)\,du$ where $u = \sqrt[n]{ax+b}$

16.52 $\int F(\sqrt{a^2-x^2})\,dx = a\int F(a\cos u)\cos u\,du$ where $x = a\sin u$

16.53 $\int F(\sqrt{x^2+a^2})\,dx = a\int F(a\sec u)\sec^2 u\,du$ where $x = a\tan u$

16.54 $\int F(\sqrt{x^2-a^2})\,dx = a\int F(a\tan u)\sec u\tan u\,du$ where $x = a\sec u$

16.55 $\int F(e^{ax})\,dx = \frac{1}{a}\int \frac{F(u)}{u}\,du$ where $u = e^{ax}$

16.56 $\int F(\ln x)\,dx = \int F(u)\,e^u\,du$ where $u = \ln x$

16.57 $\int F\left(\sin^{-1}\frac{x}{a}\right)dx = a\int F(u)\cos u\,du$ where $u = \sin^{-1}\frac{x}{a}$

Similar results apply for other inverse trigonometric functions.

16.58 $\int F(\sin x, \cos x)\,dx = 2\int F\left(\frac{2u}{1+u^2}, \frac{1-u^2}{1+u^2}\right)\frac{du}{1+u^2}$ where $u = \tan\frac{x}{2}$

17 TABLES OF SPECIAL INDEFINITE INTEGRALS

Here we provide tables of special indefinite integrals. As stated in the remarks on page 64, here a, b, p, q, n are constants, restricted if indicated; $e = 2.71828 \ldots$ is the natural base of logarithms; $\ln u$ denotes the natural logarithm of u, where it is assumed that $u > 0$ [in general, to extend formulas to cases where $u < 0$ as well, replace $\ln u$ by $\ln|u|$]; all angles are in radians; and all constants of integration are omitted but implied. It is assumed in all cases that division by zero is excluded.

Our integrals are divided into types which involve the following algebraic expressions and functions:

(1) $ax+b$
(2) $\sqrt{ax+b}$
(3) $ax+b$ and $px+q$
(4) $\sqrt{ax+b}$ and $px+q$
(5) $\sqrt{ax+b}$ and $\sqrt{px+q}$
(6) x^2+a^2
(7) x^2-a^2, with $x^2>a^2$
(8) a^2-x^2, with $x^2<a^2$
(9) $\sqrt{x^2+a^2}$
(10) $\sqrt{x^2-a^2}$
(11) $\sqrt{a^2-x^2}$
(12) ax^2+bx+c
(13) $\sqrt{ax^2+bx+c}$
(14) x^3+a^3
(15) $x^4 \pm a^4$
(16) $x^n \pm a^n$
(17) $\sin ax$
(18) $\cos ax$
(19) $\sin ax$ and $\cos ax$
(20) $\tan ax$
(21) $\cot ax$
(22) $\sec ax$
(23) $\csc ax$
(24) inverse trigonometric functions
(25) e^{ax}
(26) $\ln x$
(27) $\sinh ax$
(28) $\cosh ax$
(29) $\sinh ax$ and $\cosh ax$
(30) $\tanh ax$
(31) $\coth ax$
(32) $\operatorname{sech} ax$
(33) $\operatorname{csch} ax$
(34) inverse hyperbolic functions

Some integrals contain the Bernouilli numbers B_n and the Euler numbers E_n defined in Chapter 23 (pages 139–140).

(1) INTEGRALS INVOLVING $ax+b$

17.1.1 $$\int \frac{dx}{ax+b} = \frac{1}{a}\ln(ax+b)$$

17.1.2 $$\int \frac{x\,dx}{ax+b} = \frac{x}{a} - \frac{b}{a^2}\ln(ax+b)$$

17.1.3 $$\int \frac{x^2\,dx}{ax+b} = \frac{(ax+b)^2}{2a^3} - \frac{2b(ax+b)}{a^3} + \frac{b^2}{a^3}\ln(ax+b)$$

17.1.4 $$\int \frac{dx}{x(ax+b)} = \frac{1}{b}\ln\left(\frac{x}{ax+b}\right)$$

17.1.5 $$\int \frac{dx}{x^2(ax+b)} = -\frac{1}{bx} + \frac{a}{b^2}\ln\left(\frac{ax+b}{x}\right)$$

17.1.6 $$\int \frac{dx}{(ax+b)^2} = \frac{-1}{a(ax+b)}$$

17.1.7 $$\int \frac{x\,dx}{(ax+b)^2} = \frac{b}{a^2(ax+b)} + \frac{1}{a^2}\ln(ax+b)$$

17.1.8 $$\int \frac{x^2\,dx}{(ax+b)^2} = \frac{ax+b}{a^3} - \frac{b^2}{a^3(ax+b)} - \frac{2b}{a^3}\ln(ax+b)$$

17.1.9 $$\int \frac{dx}{x(ax+b)^2} = \frac{1}{b(ax+b)} + \frac{1}{b^2}\ln\left(\frac{x}{ax+b}\right)$$

17.1.10 $$\int \frac{dx}{x^2(ax+b)^2} = \frac{-a}{b^2(ax+b)} - \frac{1}{b^2x} + \frac{2a}{b^3}\ln\left(\frac{ax+b}{x}\right)$$

17.1.11 $$\int \frac{dx}{(ax+b)^3} = \frac{-1}{2(ax+b)^2}$$

17.1.12 $$\int \frac{x\,dx}{(ax+b)^3} = \frac{-1}{a^2(ax+b)} + \frac{b}{2a^2(ax+b)^2}$$

17.1.13 $$\int \frac{x^2\,dx}{(ax+b)^3} = \frac{2b}{a^3(ax+b)} - \frac{b^2}{2a^3(ax+b)^2} + \frac{1}{a^3}\ln(ax+b)$$

17.1.14 $$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{(n+1)a}. \quad \text{If } n=-1, \text{ see 17.1.1.}$$

17.1.15 $$\int x(ax+b)^n\,dx = \frac{(ax+b)^{n+2}}{(n+2)a^2} - \frac{b(ax+b)^{n+1}}{(n+1)a^2}, \quad n \neq -1, -2$$

If $n = -1, -2$, see 17.1.2 and 17.1.7.

17.1.16 $$\int x^2(ax+b)^n\,dx = \frac{(ax+b)^{n+3}}{(n+3)a^3} - \frac{2b(ax+b)^{n+2}}{(n+2)a^3} + \frac{b^2(ax+b)^{n+1}}{(n+1)a^3}$$

If $n = -1, -2, -3$, see 17.1.3, 17.1.8, and 17.1.13.

17.1.17 $$\int x^m(ax+b)^n\,dx = \begin{cases} \dfrac{x^{m+1}(ax+b)^n}{m+n+1} + \dfrac{nb}{m+n+1}\displaystyle\int x^m(ax+b)^{n-1}\,dx \\ \dfrac{x^m(ax+b)^{n+1}}{(m+n+1)a} - \dfrac{mb}{(m+n+1)a}\displaystyle\int x^{m-1}(ax+b)^n\,dx \\ \dfrac{-x^{m+1}(ax+b)^{n+1}}{(n+1)b} + \dfrac{m+n+2}{(n+1)b}\displaystyle\int x^m(ax+b)^{n+1}\,dx \end{cases}$$

(2) INTEGRALS INVOLVING $\sqrt{ax+b}$

17.2.1 $$\int \frac{dx}{\sqrt{ax+b}} = \frac{2\sqrt{ax+b}}{a}$$

17.2.2 $$\int \frac{x\,dx}{\sqrt{ax+b}} = \frac{2(ax-2b)}{3a^2}\sqrt{ax+b}$$

17.2.3 $$\int \frac{x^2\,dx}{\sqrt{ax+b}} = \frac{2(3a^2x^2-4abx+8b^2)}{15a^3}\sqrt{ax+b}$$

17.2.4 $$\int \frac{dx}{x\sqrt{ax+b}} = \begin{cases} \dfrac{1}{\sqrt{b}}\ln\left(\dfrac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right) \\ \dfrac{2}{\sqrt{-b}}\tan^{-1}\sqrt{\dfrac{ax+b}{-b}} \end{cases}$$

17.2.5 $$\int \frac{dx}{x^2\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{bx} - \frac{a}{2b}\int \frac{dx}{x\sqrt{ax+b}} \quad \text{[See 17.2.12].}$$

17.2.6 $$\int \sqrt{ax+b}\,dx = \frac{2\sqrt{(ax+b)^3}}{3a}$$

17.2.7 $\displaystyle\int x\sqrt{ax+b}\,dx = \frac{2(3ax-2b)}{15a^2}\sqrt{(ax+b)^3}$

17.2.8 $\displaystyle\int x^2\sqrt{ax+b}\,dx = \frac{2(15a^2x^2-12abx+8b^2)}{105a^3}\sqrt{(ax+b)^3}$

17.2.9 $\displaystyle\int \frac{\sqrt{ax+b}}{x}\,dx = 2\sqrt{ax+b} + b\int \frac{dx}{x\sqrt{ax+b}}$ [See 17.2.12]

17.2.10 $\displaystyle\int \frac{\sqrt{ax+b}}{x^2}\,dx = -\frac{\sqrt{ax+b}}{x} + \frac{a}{2}\int \frac{dx}{x\sqrt{ax+b}}$ [See 17.2.12]

17.2.11 $\displaystyle\int \frac{x^m}{\sqrt{ax+b}}\,dx = \frac{2x^m\sqrt{ax+b}}{(2m+1)a} - \frac{2mb}{(2m+1)a}\int \frac{x^{m-1}}{\sqrt{ax+b}}\,dx$

17.2.12 $\displaystyle\int \frac{dx}{x^m\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{(m-1)bx^{m-1}} - \frac{(2m-3)a}{(2m-2)b}\int \frac{dx}{x^{m-1}\sqrt{ax+b}}$

17.2.13 $\displaystyle\int x^m\sqrt{ax+b}\,dx = \frac{2x^m}{(2m+3)a}(ax+b)^{3/2} - \frac{2mb}{(2m+3)a}\int x^{m-1}\sqrt{ax+b}\,dx$

17.2.14 $\displaystyle\int \frac{\sqrt{ax+b}}{x^m}\,dx = -\frac{\sqrt{ax+b}}{(m-1)x^{m-1}} + \frac{a}{2(m-1)}\int \frac{dx}{x^{m-1}\sqrt{ax+b}}$

17.2.15 $\displaystyle\int \frac{\sqrt{ax+b}}{x^m}\,dx = \frac{-(ax+b)^{3/2}}{(m-1)bx^{m-1}} - \frac{(2m-5)a}{(2m-2)b}\int \frac{\sqrt{ax+b}}{x^{m-1}}\,dx$

17.2.16 $\displaystyle\int (ax+b)^{m/2}\,dx = \frac{2(ax+b)^{(m+2)/2}}{a^2(m+2)}$

17.2.17 $\displaystyle\int x(ax+b)^{m/2}\,dx = \frac{2(ax+b)^{(m+4)/2}}{a^2(m+4)} - \frac{2b(ax+b)^{(m+2)/2}}{a^2(m+2)}$

17.2.18 $\displaystyle\int x^2(ax+b)^{m/2}\,dx = \frac{2(ax+b)^{(m+6)/2}}{a^3(m+6)} - \frac{4b(ax+b)^{(m+4)/2}}{a^3(m+4)} + \frac{2b^2(ax+b)^{(m+2)/2}}{a^3(m+2)}$

17.2.19 $\displaystyle\int \frac{(ax+b)^{m/2}}{x}\,dx = \frac{2(ax+b)^{m/2}}{m} + b\int \frac{(ax+b)^{(m-2)/2}}{x}\,dx$

17.2.20 $\displaystyle\int \frac{(ax+b)^{m/2}}{x^2}\,dx = -\frac{(ax+b)^{(m+2)/2}}{bx} + \frac{ma}{2b}\int \frac{(ax+b)^{m/2}}{x}\,dx$

17.2.21 $\displaystyle\int \frac{dx}{x(ax+b)^{m/2}} = \frac{2}{(m-2)b(ax+b)^{(m-2)/2}} + \frac{1}{b}\int \frac{dx}{x(ax+b)^{(m-2)/2}}$

[illegible]

17.3.1 $\displaystyle\int \frac{dx}{(ax+b)(px+q)} = \frac{1}{bp-aq}\ln\left(\frac{px+q}{ax+b}\right)$

17.3.2 $\displaystyle\int \frac{x\,dx}{(ax+b)(px+q)} = \frac{1}{bp-aq}\left\{\frac{b}{a}\ln(ax+b) - \frac{q}{p}\ln(px+q)\right\}$

17.3.3 $$\int \frac{dx}{(ax+b)^2(px+q)} = \frac{1}{bp-aq}\left\{\frac{1}{ax+b} + \frac{p}{bp-aq}\ln\left(\frac{px+q}{ax+b}\right)\right\}$$

17.3.4 $$\int \frac{x\,dx}{(ax+b)^2(px+q)} = \frac{1}{bp-aq}\left\{\frac{q}{bp-aq}\ln\left(\frac{ax+b}{px+q}\right) - \frac{b}{a(ax+b)}\right\}$$

17.3.5 $$\int \frac{x^2\,dx}{(ax+b)^2(px+q)} = \frac{b^2}{(bp-aq)a^2(ax+b)} + \frac{1}{(bp-aq)^2}\left\{\frac{q^2}{p}\ln(px+q) + \frac{b(bp-2aq)}{a^2}\ln(ax+b)\right\}$$

17.3.6 $$\int \frac{dx}{(ax+b)^m(px+q)^n} = \frac{-1}{(n-1)(bp-aq)}\left\{\frac{1}{(ax+b)^{m-1}(px+q)^{n-1}} + a(m+n-2)\int \frac{dx}{(ax+b)^m(px+q)^{n-1}}\right\}$$

17.3.7 $$\int \frac{ax+b}{px+q}\,dx = \frac{ax}{p} + \frac{bp-aq}{p^2}\ln(px+q)$$

17.3.8 $$\int \frac{(ax+b)^m}{(px+q)^n}\,dx = \begin{cases} \dfrac{-1}{(n-1)(bp-aq)}\left\{\dfrac{(ax+b)^{m+1}}{(px+q)^{n-1}} + (n-m-2)a\displaystyle\int \dfrac{(ax+b)^m}{(px+q)^{n-1}}\,dx\right\} \\ \dfrac{-1}{(n-m-1)p}\left\{\dfrac{(ax+b)^m}{(px+q)^{n-1}} + m(bp-aq)\displaystyle\int \dfrac{(ax+b)^{m-1}}{(px+q)^n}\,dx\right\} \\ \dfrac{-1}{(n-1)p}\left\{\dfrac{(ax+b)^m}{(px+q)^{n-1}} - ma\displaystyle\int \dfrac{(ax+b)^{m-1}}{(px+q)^{n-1}}\,dx\right\} \end{cases}$$

(4) INTEGRALS INVOLVING $\sqrt{ax+b}$ AND $px+q$

17.4.1 $$\int \frac{px+q}{\sqrt{ax+b}}\,dx = \frac{2(apx+3aq-2bp)}{3a^2}\sqrt{ax+b}$$

17.4.2 $$\int \frac{dx}{(px+q)\sqrt{ax+b}} = \begin{cases} \dfrac{1}{\sqrt{bp-aq}\sqrt{p}}\ln\left(\dfrac{\sqrt{p(ax+b)}-\sqrt{bp-aq}}{\sqrt{p(ax+b)}+\sqrt{bp-aq}}\right) \\ \dfrac{2}{\sqrt{aq-bp}\sqrt{p}}\tan^{-1}\sqrt{\dfrac{p(ax+b)}{aq-bp}} \end{cases}$$

17.4.3 $$\int \frac{\sqrt{ax+b}}{px+q}\,dx = \begin{cases} \dfrac{2\sqrt{ax+b}}{p} + \dfrac{\sqrt{bp-aq}}{p\sqrt{p}}\ln\left(\dfrac{\sqrt{p(ax+b)}-\sqrt{bp-aq}}{\sqrt{p(ax+b)}+\sqrt{bp-aq}}\right) \\ \dfrac{2\sqrt{ax+b}}{p} - \dfrac{2\sqrt{aq-bp}}{p\sqrt{p}}\tan^{-1}\sqrt{\dfrac{p(ax+b)}{aq-bp}} \end{cases}$$

17.4.4 $$\int (px+q)^n\sqrt{ax+b}\,dx = \frac{2(px+q)^{n+1}\sqrt{ax+b}}{(2n+3)p} + \frac{bp-aq}{(2n+3)p}\int \frac{(px+q)^n}{\sqrt{ax+b}}\,dx$$

17.4.5 $$\int \frac{dx}{(px+q)^n\sqrt{ax+b}} = \frac{\sqrt{ax+b}}{(n-1)(aq-bp)(px+q)^{n-1}} + \frac{(2n-3)a}{2(n-1)(aq-bp)}\int \frac{dx}{(px+q)^{n-1}\sqrt{ax+b}}$$

17.4.6 $$\int \frac{(px+q)^n}{\sqrt{ax+b}}\,dx = \frac{2(px+q)^n\sqrt{ax+b}}{(2n+1)a} + \frac{2n(aq-bp)}{(2n+1)a}\int \frac{(px+q)^{n-1}\,dx}{\sqrt{ax+b}}$$

17.4.7 $$\int \frac{\sqrt{ax+b}}{(px+q)^n}\,dx = \frac{-\sqrt{ax+b}}{(n-1)p(px+q)^{n-1}} + \frac{a}{2(n-1)p}\int \frac{dx}{(px+q)^{n-1}\sqrt{ax+b}}$$

(5) INTEGRALS INVOLVING $\sqrt{ax+b}$ AND $\sqrt{px+q}$

17.5.1 $$\int \frac{dx}{\sqrt{(ax+b)(px+q)}} = \begin{cases} \dfrac{2}{\sqrt{ap}} \ln\left(\sqrt{a(px+q)} + \sqrt{p(ax+b)}\right) \\ \dfrac{2}{\sqrt{-ap}} \tan^{-1} \sqrt{\dfrac{-p(ax+b)}{a(px+q)}} \end{cases}$$

17.5.2 $$\int \frac{x\,dx}{\sqrt{(ax+b)(px+q)}} = \frac{\sqrt{(ax+b)(px+q)}}{ap} - \frac{bp+aq}{2ap} \int \frac{dx}{\sqrt{(ax+b)(px+q)}}$$

17.5.3 $$\int \sqrt{(ax+b)(px+q)}\,dx = \frac{2apx+bp+aq}{4ap}\sqrt{(ax+b)(px+q)} - \frac{(bp-aq)^2}{8ap} \int \frac{dx}{\sqrt{(ax+b)(px+q)}}$$

17.5.4 $$\int \sqrt{\frac{px+q}{ax+b}}\,dx = \frac{\sqrt{(ax+b)(px+q)}}{a} + \frac{aq-bp}{2a} \int \frac{dx}{\sqrt{(ax+b)(px+q)}}$$

17.5.5 $$\int \frac{dx}{(px+q)\sqrt{(ax+b)(px+q)}} = \frac{2\sqrt{ax+b}}{(aq-bp)\sqrt{px+q}}$$

(6) INTEGRALS INVOLVING x^2+a^2

17.6.1 $$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

17.6.2 $$\int \frac{x\,dx}{x^2+a^2} = \frac{1}{2}\ln(x^2+a^2)$$

17.6.3 $$\int \frac{x^2\,dx}{x^2+a^2} = x - a\tan^{-1}\frac{x}{a}$$

17.6.4 $$\int \frac{x^3\,dx}{x^2+a^2} = \frac{x^2}{2} - \frac{a^2}{2}\ln(x^2+a^2)$$

17.6.5 $$\int \frac{dx}{x(x^2+a^2)} = \frac{1}{2a^2}\ln\left(\frac{x^2}{x^2+a^2}\right)$$

17.6.6 $$\int \frac{dx}{x^2(x^2+a^2)} = -\frac{1}{a^2x} - \frac{1}{a^3}\tan^{-1}\frac{x}{a}$$

17.6.7 $$\int \frac{dx}{x^3(x^2+a^2)} = -\frac{1}{2a^2x^2} - \frac{1}{2a^4}\ln\left(\frac{x^2}{x^2+a^2}\right)$$

17.6.8 $$\int \frac{dx}{(x^2+a^2)^2} = \frac{x}{2a^2(x^2+a^2)} + \frac{1}{2a^3}\tan^{-1}\frac{x}{a}$$

17.6.9 $$\int \frac{x\,dx}{(x^2+a^2)^2} = \frac{-1}{2(x^2+a^2)}$$

17.6.10 $$\int \frac{x^2\,dx}{(x^2+a^2)^2} = \frac{-x}{2(x^2+a^2)} + \frac{1}{2a}\tan^{-1}\frac{x}{a}$$

17.6.11 $\int \frac{x^3\,dx}{(x^2+a^2)^2} = \frac{a^2}{2(x^2+a^2)} + \frac{1}{2}\ln(x^2+a^2)$

17.6.12 $\int \frac{dx}{x(x^2+a^2)^2} = \frac{1}{2a^2(x^2+a^2)} + \frac{1}{2a^4}\ln\left(\frac{x^2}{x^2+a^2}\right)$

17.6.13 $\int \frac{dx}{x^2(x^2+a^2)^2} = -\frac{1}{a^4x} - \frac{x}{2a^4(x^2+a^2)} - \frac{3}{2a^5}\tan^{-1}\frac{x}{a}$

17.6.14 $\int \frac{dx}{x^3(x^2+a^2)^2} = -\frac{1}{2a^4x^2} - \frac{1}{2a^4(x^2+a^2)} - \frac{1}{a^6}\ln\left(\frac{x^2}{x^2+a^2}\right)$

17.6.15 $\int \frac{dx}{(x^2+a^2)^n} = \frac{x}{2(n-1)a^2(x^2+a^2)^{n-1}} + \frac{2n-3}{(2n-2)a^2}\int \frac{dx}{(x^2+a^2)^{n-1}}$

17.6.16 $\int \frac{x\,dx}{(x^2+a^2)^n} = \frac{-1}{2(n-1)(x^2+a^2)^{n-1}}$

17.6.17 $\int \frac{dx}{x(x^2+a^2)^n} = \frac{1}{2(n-1)a^2(x^2+a^2)^{n-1}} + \frac{1}{a^2}\int \frac{dx}{x(x^2+a^2)^{n-1}}$

17.6.18 $\int \frac{x^m\,dx}{(x^2+a^2)^n} = \int \frac{x^{m-2}\,dx}{(x^2+a^2)^{n-1}} - a^2\int \frac{x^{m-2}\,dx}{(x^2+a^2)^n}$

17.6.19 $\int \frac{dx}{x^m(x^2+a^2)^n} = \frac{1}{a^2}\int \frac{dx}{x^m(x^2+a^2)^{n-1}} - \frac{1}{a^2}\int \frac{dx}{x^{m-2}(x^2+a^2)^n}$

17.7 INTEGRALS INVOLVING $x^2 - a^2$, $x^2 > a^2$

17.7.1 $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right)$ or $-\frac{1}{a}\coth^{-1}\frac{x}{a}$

17.7.2 $\int \frac{x\,dx}{x^2-a^2} = \frac{1}{2}\ln(x^2-a^2)$

17.7.3 $\int \frac{x^2\,dx}{x^2-a^2} = x + \frac{a}{2}\ln\left(\frac{x-a}{x+a}\right)$

17.7.4 $\int \frac{x^3\,dx}{x^2-a^2} = \frac{x^2}{2} + \frac{a^2}{2}\ln(x^2-a^2)$

17.7.5 $\int \frac{dx}{x(x^2-a^2)} = \frac{1}{2a^2}\ln\left(\frac{x^2-a^2}{x^2}\right)$

17.7.6 $\int \frac{dx}{x^2(x^2-a^2)} = \frac{1}{a^2x} + \frac{1}{2a^3}\ln\left(\frac{x-a}{x+a}\right)$

17.7.7 $\int \frac{dx}{x^3(x^2-a^2)} = \frac{1}{2a^2x^2} - \frac{1}{2a^4}\ln\left(\frac{x^2}{x^2-a^2}\right)$

17.7.8 $\int \frac{dx}{(x^2-a^2)^2} = \frac{-x}{2a^2(x^2-a^2)} - \frac{1}{4a^3}\ln\left(\frac{x-a}{x+a}\right)$

17.7.9 $\displaystyle\int \frac{x\,dx}{(x^2-a^2)^2} = \frac{-1}{2(x^2-a^2)}$

17.7.10 $\displaystyle\int \frac{x^2\,dx}{(x^2-a^2)^2} = \frac{-x}{2(x^2-a^2)} + \frac{1}{4a}\ln\left(\frac{x-a}{x+a}\right)$

17.7.11 $\displaystyle\int \frac{x^3\,dx}{(x^2-a^2)^2} = \frac{-a^2}{2(x^2-a^2)} + \frac{1}{2}\ln(x^2-a^2)$

17.7.12 $\displaystyle\int \frac{dx}{x(x^2-a^2)^2} = \frac{-1}{2a^2(x^2-a^2)} + \frac{1}{2a^4}\ln\left(\frac{x^2}{x^2-a^2}\right)$

17.7.13 $\displaystyle\int \frac{dx}{x^2(x^2-a^2)^2} = -\frac{1}{a^4x} - \frac{x}{2a^4(x^2-a^2)} - \frac{3}{4a^5}\ln\left(\frac{x-a}{x+a}\right)$

17.7.14 $\displaystyle\int \frac{dx}{x^3(x^2-a^2)^2} = -\frac{1}{2a^4x^2} - \frac{1}{2a^4(x^2-a^2)} + \frac{1}{a^6}\ln\left(\frac{x^2}{x^2-a^2}\right)$

17.7.15 $\displaystyle\int \frac{dx}{(x^2-a^2)^n} = \frac{-x}{2(n-1)a^2(x^2-a^2)^{n-1}} - \frac{2n-3}{(2n-2)a^2}\int \frac{dx}{(x^2-a^2)^{n-1}}$

17.7.16 $\displaystyle\int \frac{x\,dx}{(x^2-a^2)^n} = \frac{-1}{2(n-1)(x^2-a^2)^{n-1}}$

17.7.17 $\displaystyle\int \frac{dx}{x(x^2-a^2)^n} = \frac{-1}{2(n-1)a^2(x^2-a^2)^{n-1}} - \frac{1}{a^2}\int \frac{dx}{x(x^2-a^2)^{n-1}}$

17.7.18 $\displaystyle\int \frac{x^m\,dx}{(x^2-a^2)^n} = \int \frac{x^{m-2}\,dx}{(x^2-a^2)^{n-1}} + a^2\int \frac{x^{m-2}\,dx}{(x^2-a^2)^n}$

17.7.19 $\displaystyle\int \frac{dx}{x^m(x^2-a^2)^n} = \frac{1}{a^2}\int \frac{dx}{x^{m-2}(x^2-a^2)^n} - \frac{1}{a^2}\int \frac{dx}{x^m(x^2-a^2)^{n-1}}$

(8) INTEGRALS INVOLVING a^2-x^2, $x^2<a^2$

17.8.1 $\displaystyle\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right)$ or $\displaystyle\frac{1}{a}\tanh^{-1}\frac{x}{a}$

17.8.2 $\displaystyle\int \frac{x\,dx}{a^2-x^2} = -\frac{1}{2}\ln(a^2-x^2)$

17.8.3 $\displaystyle\int \frac{x^2\,dx}{a^2-x^2} = -x + \frac{a}{2}\ln\left(\frac{a+x}{a-x}\right)$

17.8.4 $\displaystyle\int \frac{x^3\,dx}{a^2-x^2} = -\frac{x^2}{2} - \frac{a^2}{2}\ln(a^2-x^2)$

17.8.5 $\displaystyle\int \frac{dx}{x(a^2-x^2)} = \frac{1}{2a^2}\ln\left(\frac{x^2}{a^2-x^2}\right)$

17.8.6 $\displaystyle\int \frac{dx}{x^2(a^2-x^2)} = -\frac{1}{a^2x} + \frac{1}{2a^3}\ln\left(\frac{a+x}{a-x}\right)$

17.8.7 $$\int \frac{dx}{x^3(a^2-x^2)} = -\frac{1}{2a^2x^2} + \frac{1}{2a^4}\ln\left(\frac{x^2}{a^2-x^2}\right)$$

17.8.8 $$\int \frac{dx}{(a^2-x^2)^2} = \frac{x}{2a^2(a^2-x^2)} + \frac{1}{4a^3}\ln\left(\frac{a+x}{a-x}\right)$$

17.8.9 $$\int \frac{x\,dx}{(a^2-x^2)^2} = \frac{1}{2(a^2-x^2)}$$

17.8.10 $$\int \frac{x^2\,dx}{(a^2-x^2)^2} = \frac{x}{2(a^2-x^2)} - \frac{1}{4a}\ln\left(\frac{a+x}{a-x}\right)$$

17.8.11 $$\int \frac{x^3\,dx}{(a^2-x^2)^2} = \frac{a^2}{2(a^2-x^2)} + \frac{1}{2}\ln(a^2-x^2)$$

17.8.12 $$\int \frac{dx}{x(a^2-x^2)^2} = \frac{1}{2a^2(a^2-x^2)} + \frac{1}{2a^4}\ln\left(\frac{x^2}{a^2-x^2}\right)$$

17.8.13 $$\int \frac{dx}{x^2(a^2-x^2)^2} = \frac{-1}{a^4x} + \frac{x}{2a^4(a^2-x^2)} + \frac{3}{4a^5}\ln\left(\frac{a+x}{a-x}\right)$$

17.8.14 $$\int \frac{dx}{x^3(a^2-x^2)^2} = \frac{-1}{2a^4x^2} + \frac{1}{2a^4(a^2-x^2)} + \frac{1}{a^6}\ln\left(\frac{x^2}{a^2-x^2}\right)$$

17.8.15 $$\int \frac{dx}{(a^2-x^2)^n} = \frac{x}{2(n-1)a^2(a^2-x^2)^{n-1}} + \frac{2n-3}{(2n-2)a^2}\int \frac{dx}{(a^2-x^2)^{n-1}}$$

17.8.16 $$\int \frac{x\,dx}{(a^2-x^2)^n} = \frac{1}{2(n-1)(a^2-x^2)^{n-1}}$$

17.8.17 $$\int \frac{dx}{x(a^2-x^2)^n} = \frac{1}{2(n-1)a^2(a^2-x^2)^{n-1}} + \frac{1}{a^2}\int \frac{dx}{x(a^2-x^2)^{n-1}}$$

17.8.18 $$\int \frac{x^m\,dx}{(a^2-x^2)^n} = a^2\int \frac{x^{m-2}\,dx}{(a^2-x^2)^n} - \int \frac{x^{m-2}\,dx}{(a^2-x^2)^{n-1}}$$

17.8.19 $$\int \frac{dx}{x^m(a^2-x^2)^n} = \frac{1}{a^2}\int \frac{dx}{x^m(a^2-x^2)^{n-1}} + \frac{1}{a^2}\int \frac{dx}{x^{m-2}(a^2-x^2)^n}$$

(9) INTEGRALS INVOLVING $\sqrt{x^2+a^2}$

17.9.1 $$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln(x+\sqrt{x^2+a^2}) \quad \text{or} \quad \sinh^{-1}\frac{x}{a}$$

17.9.2 $$\int \frac{x\,dx}{\sqrt{x^2+a^2}} = \sqrt{x^2+a^2}$$

17.9.3 $$\int \frac{x^2\,dx}{\sqrt{x^2+a^2}} = \frac{x\sqrt{x^2+a^2}}{2} - \frac{a^2}{2}\ln(x+\sqrt{x^2+a^2})$$

17.9.4 $$\int \frac{x^3\,dx}{\sqrt{x^2+a^2}} = \frac{(x^2+a^2)^{3/2}}{3} - a^2\sqrt{x^2+a^2}$$

17.9.5 $$\int \frac{dx}{x\sqrt{x^2+a^2}} = -\frac{1}{a}\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$$

17.9.6 $$\int \frac{dx}{x^2\sqrt{x^2+a^2}} = -\frac{\sqrt{x^2+a^2}}{a^2x}$$

17.9.7 $$\int \frac{dx}{x^3\sqrt{x^2+a^2}} = -\frac{\sqrt{x^2+a^2}}{2a^2x^2} + \frac{1}{2a^3}\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$$

17.9.8 $$\int \sqrt{x^2+a^2}\,dx = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\ln(x+\sqrt{x^2+a^2})$$

17.9.9 $$\int x\sqrt{x^2+a^2}\,dx = \frac{(x^2+a^2)^{3/2}}{3}$$

17.9.10 $$\int x^2\sqrt{x^2+a^2}\,dx = \frac{x(x^2+a^2)^{3/2}}{4} - \frac{a^2x\sqrt{x^2+a^2}}{8} - \frac{a^4}{8}\ln(x+\sqrt{x^2+a^2})$$

17.9.11 $$\int x^3\sqrt{x^2+a^2}\,dx = \frac{(x^2+a^2)^{5/2}}{5} - \frac{a^2(x^2+a^2)^{3/2}}{3}$$

17.9.12 $$\int \frac{\sqrt{x^2+a^2}}{x}\,dx = \sqrt{x^2+a^2} - a\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$$

17.9.13 $$\int \frac{\sqrt{x^2+a^2}}{x^2}\,dx = -\frac{\sqrt{x^2+a^2}}{x} + \ln(x+\sqrt{x^2+a^2})$$

17.9.14 $$\int \frac{\sqrt{x^2+a^2}}{x^3}\,dx = -\frac{\sqrt{x^2+a^2}}{2x^2} - \frac{1}{2a}\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$$

17.9.15 $$\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2\sqrt{x^2+a^2}}$$

17.9.16 $$\int \frac{x\,dx}{(x^2+a^2)^{3/2}} = \frac{-1}{\sqrt{x^2+a^2}}$$

17.9.17 $$\int \frac{x^2\,dx}{(x^2+a^2)^{3/2}} = \frac{-x}{\sqrt{x^2+a^2}} + \ln(x+\sqrt{x^2+a^2})$$

17.9.18 $$\int \frac{x^3\,dx}{(x^2+a^2)^{3/2}} = \sqrt{x^2+a^2} + \frac{a^2}{\sqrt{x^2+a^2}}$$

17.9.19 $$\int \frac{dx}{x(x^2+a^2)^{3/2}} = \frac{1}{a^2\sqrt{x^2+a^2}} - \frac{1}{a^3}\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$$

17.9.20 $$\int \frac{dx}{x^2(x^2+a^2)^{3/2}} = -\frac{\sqrt{x^2+a^2}}{a^4x} - \frac{x}{a^4\sqrt{x^2+a^2}}$$

17.9.21 $$\int \frac{dx}{x^3(x^2+a^2)^{3/2}} = \frac{-1}{2a^2x^2\sqrt{x^2+a^2}} - \frac{3}{2a^4\sqrt{x^2+a^2}} + \frac{3}{2a^5}\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$$

17.9.22 $$\int (x^2+a^2)^{3/2}\,dx = \frac{x(x^2+a^2)^{3/2}}{4} + \frac{3a^2x\sqrt{x^2+a^2}}{8} + \frac{3}{8}a^4\ln(x+\sqrt{x^2+a^2})$$

17.9.23 $$\int x(x^2+a^2)^{3/2}\,dx = \frac{(x^2+a^2)^{5/2}}{5}$$

17.9.24 $\int x^2(x^2+a^2)^{3/2}\,dx = \frac{x(x^2+a^2)^{5/2}}{6} - \frac{a^2x(x^2+a^2)^{3/2}}{24} - \frac{a^4x\sqrt{x^2+a^2}}{16} - \frac{a^6}{16}\ln(x+\sqrt{x^2+a^2})$

17.9.25 $\int x^3(x^2+a^2)^{3/2}\,dx = \frac{(x^2+a^2)^{7/2}}{7} - \frac{a^2(x^2+a^2)^{5/2}}{5}$

17.9.26 $\int \frac{(x^2+a^2)^{3/2}}{x}\,dx = \frac{(x^2+a^2)^{3/2}}{3} + a^2\sqrt{x^2+a^2} - a^3\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$

17.9.27 $\int \frac{(x^2+a^2)^{3/2}}{x^2}\,dx = -\frac{(x^2+a^2)^{3/2}}{x} + \frac{3x\sqrt{x^2+a^2}}{2} + \frac{3}{2}a^2\ln(x+\sqrt{x^2+a^2})$

17.9.28 $\int \frac{(x^2+a^2)^{3/2}}{x^3}\,dx = -\frac{(x^2+a^2)^{3/2}}{2x^2} + \frac{3}{2}\sqrt{x^2+a^2} - \frac{3}{2}a\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$

(10) INTEGRALS INVOLVING $\sqrt{x^2-a^2}$

17.10.1 $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln(x+\sqrt{x^2-a^2}), \qquad \int \frac{x\,dx}{\sqrt{x^2-a^2}} = \sqrt{x^2-a^2}$

17.10.2 $\int \frac{x^2\,dx}{\sqrt{x^2-a^2}} = \frac{x\sqrt{x^2-a^2}}{2} + \frac{a^2}{2}\ln(x+\sqrt{x^2-a^2})$

17.10.3 $\int \frac{x^3\,dx}{\sqrt{x^2-a^2}} = \frac{(x^2-a^2)^{3/2}}{3} + a^2\sqrt{x^2-a^2}$

17.10.4 $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right|$

17.10.5 $\int \frac{dx}{x^2\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{a^2x}$

17.10.6 $\int \frac{dx}{x^3\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{2a^2x^2} + \frac{1}{2a^3}\sec^{-1}\left|\frac{x}{a}\right|$

17.10.7 $\int \sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\ln(x+\sqrt{x^2-a^2})$

17.10.8 $\int x\sqrt{x^2-a^2}\,dx = \frac{(x^2-a^2)^{3/2}}{3}$

17.10.9 $\int x^2\sqrt{x^2-a^2}\,dx = \frac{x(x^2-a^2)^{3/2}}{4} + \frac{a^2x\sqrt{x^2-a^2}}{8} - \frac{a^4}{8}\ln(x+\sqrt{x^2-a^2})$

17.10.10 $\int x^3\sqrt{x^2-a^2}\,dx = \frac{(x^2-a^2)^{5/2}}{5} + \frac{a^2(x^2-a^2)^{3/2}}{3}$

17.10.11 $\int \frac{\sqrt{x^2-a^2}}{x}\,dx = \sqrt{x^2-a^2} - a\sec^{-1}\left|\frac{x}{a}\right|$

17.10.12 $\int \frac{\sqrt{x^2-a^2}}{x^2}\,dx = -\frac{\sqrt{x^2-a^2}}{x} + \ln(x+\sqrt{x^2-a^2})$

17.10.13 $$\int \frac{\sqrt{x^2-a^2}}{x^3}\,dx = -\frac{\sqrt{x^2-a^2}}{2x^2} + \frac{1}{2a}\sec^{-1}\left|\frac{x}{a}\right|$$

17.10.14 $$\int \frac{dx}{(x^2-a^2)^{3/2}} = -\frac{x}{a^2\sqrt{x^2-a^2}}$$

17.10.15 $$\int \frac{x\,dx}{(x^2-a^2)^{3/2}} = \frac{-1}{\sqrt{x^2-a^2}}$$

17.10.16 $$\int \frac{x^2\,dx}{(x^2-a^2)^{3/2}} = -\frac{x}{\sqrt{x^2-a^2}} + \ln(x+\sqrt{x^2-a^2})$$

17.10.17 $$\int \frac{x^3\,dx}{(x^2-a^2)^{3/2}} = \sqrt{x^2-a^2} - \frac{a^2}{\sqrt{x^2-a^2}}$$

17.10.18 $$\int \frac{dx}{x(x^2-a^2)^{3/2}} = \frac{-1}{a^2\sqrt{x^2-a^2}} - \frac{1}{a^3}\sec^{-1}\left|\frac{x}{a}\right|$$

17.10.19 $$\int \frac{dx}{x^2(x^2-a^2)^{3/2}} = -\frac{\sqrt{x^2-a^2}}{a^4x} - \frac{x}{a^4\sqrt{x^2-a^2}}$$

17.10.20 $$\int \frac{dx}{x^3(x^2-a^2)^{3/2}} = \frac{1}{2a^2x^2\sqrt{x^2-a^2}} - \frac{3}{2a^4\sqrt{x^2-a^2}} - \frac{3}{2a^5}\sec^{-1}\left|\frac{x}{a}\right|$$

17.10.21 $$\int (x^2-a^2)^{3/2}\,dx = \frac{x(x^2-a^2)^{3/2}}{4} - \frac{3a^2x\sqrt{x^2-a^2}}{8} + \frac{3}{8}a^4\ln(x+\sqrt{x^2-a^2})$$

17.10.22 $$\int x(x^2-a^2)^{3/2}\,dx = \frac{(x^2-a^2)^{5/2}}{5}$$

17.10.23 $$\int x^2(x^2-a^2)^{3/2}\,dx = \frac{x(x^2-a^2)^{5/2}}{6} + \frac{a^2x(x^2-a^2)^{3/2}}{24} - \frac{a^4x\sqrt{x^2-a^2}}{16} + \frac{a^6}{16}\ln(x+\sqrt{x^2-a^2})$$

17.10.24 $$\int x^3(x^2-a^2)^{3/2}\,dx = \frac{(x^2-a^2)^{7/2}}{7} + \frac{a^2(x^2-a^2)^{5/2}}{5}$$

17.10.25 $$\int \frac{(x^2-a^2)^{3/2}}{x}\,dx = \frac{(x^2-a^2)^{3/2}}{3} - a^2\sqrt{x^2-a^2} + a^3\sec^{-1}\left|\frac{x}{a}\right|$$

17.10.26 $$\int \frac{(x^2-a^2)^{3/2}}{x^2}\,dx = -\frac{(x^2-a^2)^{3/2}}{x} + \frac{3x\sqrt{x^2-a^2}}{2} - \frac{3}{2}a^2\ln(x+\sqrt{x^2-a^2})$$

17.10.27 $$\int \frac{(x^2-a^2)^{3/2}}{x^3}\,dx = -\frac{(x^2-a^2)^{3/2}}{2x^2} + \frac{3\sqrt{x^2-a^2}}{2} - \frac{3}{2}a\sec^{-1}\left|\frac{x}{a}\right|$$

(11) INTEGRALS INVOLVING $\sqrt{a^2-x^2}$

17.11.1 $$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$$

17.11.2 $$\int \frac{x\,dx}{\sqrt{a^2-x^2}} = -\sqrt{a^2-x^2}$$

17.11.3 $\displaystyle\int \frac{x^2\,dx}{\sqrt{a^2-x^2}} = -\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$

17.11.4 $\displaystyle\int \frac{x^3\,dx}{\sqrt{a^2-x^2}} = \frac{(a^2-x^2)^{3/2}}{3} - a^2\sqrt{a^2-x^2}$

17.11.5 $\displaystyle\int \frac{dx}{x\sqrt{a^2-x^2}} = -\frac{1}{a}\ln\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)$

17.11.6 $\displaystyle\int \frac{dx}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2x}$

17.11.7 $\displaystyle\int \frac{dx}{x^3\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{2a^2x^2} - \frac{1}{2a^3}\ln\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)$

17.11.8 $\displaystyle\int \sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$

17.11.9 $\displaystyle\int x\sqrt{a^2-x^2}\,dx = -\frac{(a^2-x^2)^{3/2}}{3}$

17.11.10 $\displaystyle\int x^2\sqrt{a^2-x^2}\,dx = -\frac{x(a^2-x^2)^{3/2}}{4} + \frac{a^2x\sqrt{a^2-x^2}}{8} + \frac{a^4}{8}\sin^{-1}\frac{x}{a}$

17.11.11 $\displaystyle\int x^3\sqrt{a^2-x^2}\,dx = \frac{(a^2-x^2)^{5/2}}{5} - \frac{a^2(a^2-x^2)^{3/2}}{3}$

17.11.12 $\displaystyle\int \frac{\sqrt{a^2-x^2}}{x}\,dx = \sqrt{a^2-x^2} - a\ln\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)$

17.11.13 $\displaystyle\int \frac{\sqrt{a^2-x^2}}{x^2}\,dx = -\frac{\sqrt{a^2-x^2}}{x} - \sin^{-1}\frac{x}{a}$

17.11.14 $\displaystyle\int \frac{\sqrt{a^2-x^2}}{x^3}\,dx = -\frac{\sqrt{a^2-x^2}}{2x^2} + \frac{1}{2a}\ln\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)$

17.11.15 $\displaystyle\int \frac{dx}{(a^2-x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2-x^2}}$

17.11.16 $\displaystyle\int \frac{x\,dx}{(a^2-x^2)^{3/2}} = \frac{1}{\sqrt{a^2-x^2}}$

17.11.17 $\displaystyle\int \frac{x^2\,dx}{(a^2-x^2)^{3/2}} = \frac{x}{\sqrt{a^2-x^2}} - \sin^{-1}\frac{x}{a}$

17.11.18 $\displaystyle\int \frac{x^3\,dx}{(a^2-x^2)^{3/2}} = \sqrt{a^2-x^2} + \frac{a^2}{\sqrt{a^2-x^2}}$

17.11.19 $\displaystyle\int \frac{dx}{x(a^2-x^2)^{3/2}} = \frac{1}{a^2\sqrt{a^2-x^2}} - \frac{1}{a^3}\ln\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)$

17.11.20 $\displaystyle\int \frac{dx}{x^2(a^2-x^2)^{3/2}} = -\frac{\sqrt{a^2-x^2}}{a^4x} + \frac{x}{a^4\sqrt{a^2-x^2}}$

17.11.21 $\displaystyle\int \frac{dx}{x^3(a^2-x^2)^{3/2}} = \frac{-1}{2a^2x^2\sqrt{a^2-x^2}} + \frac{3}{2a^4\sqrt{a^2-x^2}} - \frac{3}{2a^5}\ln\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)$

17.11.22 $$\int (a^2-x^2)^{3/2}\,dx = \frac{x(a^2-x^2)^{3/2}}{4} + \frac{3a^2x\sqrt{a^2-x^2}}{8} + \frac{3}{8}a^4\sin^{-1}\frac{x}{a}$$

17.11.23 $$\int x(a^2-x^2)^{3/2}\,dx = -\frac{(a^2-x^2)^{5/2}}{5}$$

17.11.24 $$\int x^2(a^2-x^2)^{3/2}\,dx = -\frac{x(a^2-x^2)^{5/2}}{6} + \frac{a^2x(a^2-x^2)^{3/2}}{24} + \frac{a^4x\sqrt{a^2-x^2}}{16} + \frac{a^6}{16}\sin^{-1}\frac{x}{a}$$

17.11.25 $$\int x^3(a^2-x^2)^{3/2}\,dx = \frac{(a^2-x^2)^{7/2}}{7} - \frac{a^2(a^2-x^2)^{5/2}}{5}$$

17.11.26 $$\int \frac{(a^2-x^2)^{3/2}}{x}\,dx = \frac{(a^2-x^2)^{3/2}}{3} + a^2\sqrt{a^2-x^2} - a^3\ln\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)$$

17.11.27 $$\int \frac{(a^2-x^2)^{3/2}}{x^2}\,dx = -\frac{(a^2-x^2)^{3/2}}{x} - \frac{3x\sqrt{a^2-x^2}}{2} - \frac{3}{2}a^2\sin^{-1}\frac{x}{a}$$

17.11.28 $$\int \frac{(a^2-x^2)^{3/2}}{x^3}\,dx = -\frac{(a^2-x^2)^{3/2}}{2x^2} - \frac{3\sqrt{a^2-x^2}}{2} + \frac{3}{2}a\ln\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)$$

[illegible] INTEGRALS INVOLVING [illegible]

17.12.1 $$\int \frac{dx}{ax^2+bx+c} = \begin{cases} \dfrac{2}{\sqrt{4ac-b^2}}\tan^{-1}\dfrac{2ax+b}{\sqrt{4ac-b^2}} \\ \dfrac{1}{\sqrt{b^2-4ac}}\ln\left(\dfrac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right) \end{cases}$$

If $b^2 = 4ac$, $ax^2+bx+c = a(x+b/2a)^2$ and the results 17.1.6 to 17.1.10 and 17.1.14 to 17.1.17 can be used. If $b = 0$ use results on page 72. If a or $c = 0$ use results on pages 68–69.

17.12.2 $$\int \frac{x\,dx}{ax^2+bx+c} = \frac{1}{2a}\ln(ax^2+bx+c) - \frac{b}{2a}\int\frac{dx}{ax^2+bx+c}$$

17.12.3 $$\int \frac{x^2\,dx}{ax^2+bx+c} = \frac{x}{a} - \frac{b}{2a^2}\ln(ax^2+bx+c) + \frac{b^2-2ac}{2a^2}\int\frac{dx}{ax^2+bx+c}$$

17.12.4 $$\int \frac{x^m\,dx}{ax^2+bx+c} = \frac{x^{m-1}}{(m-1)a} - \frac{c}{a}\int\frac{x^{m-2}\,dx}{ax^2+bx+c} - \frac{b}{a}\int\frac{x^{m-1}\,dx}{ax^2+bx+c}$$

17.12.5 $$\int \frac{dx}{x(ax^2+bx+c)} = \frac{1}{2c}\ln\left(\frac{x^2}{ax^2+bx+c}\right) - \frac{b}{2c}\int\frac{dx}{ax^2+bx+c}$$

17.12.6 $$\int \frac{dx}{x^2(ax^2+bx+c)} = \frac{b}{2c^2}\ln\left(\frac{ax^2+bx+c}{x^2}\right) - \frac{1}{cx} + \frac{b^2-2ac}{2c^2}\int\frac{dx}{ax^2+bx+c}$$

17.12.7 $$\int \frac{dx}{x^n(ax^2+bx+c)} = -\frac{1}{(n-1)cx^{n-1}} - \frac{b}{c}\int\frac{dx}{x^{n-1}(ax^2+bx+c)} - \frac{a}{c}\int\frac{dx}{x^{n-2}(ax^2+bx+c)}$$

17.12.8 $$\int \frac{dx}{(ax^2+bx+c)^2} = \frac{2ax+b}{(4ac-b^2)(ax^2+bx+c)} + \frac{2a}{4ac-b^2}\int\frac{dx}{ax^2+bx+c}$$

17.12.9 $$\int \frac{x\,dx}{(ax^2+bx+c)^2} = -\frac{bx+2c}{(4ac-b^2)(ax^2+bx+c)} - \frac{b}{4ac-b^2}\int\frac{dx}{ax^2+bx+c}$$

17.12.10 $$\int \frac{x^2\,dx}{(ax^2+bx+c)^2} = \frac{(b^2-2ac)x+bc}{a(4ac-b^2)(ax^2+bx+c)} + \frac{2c}{4ac-b^2}\int \frac{dx}{ax^2+bx+c}$$

17.12.11 $$\int \frac{x^m\,dx}{(ax^2+bx+c)^n} = -\frac{x^{m-1}}{(2n-m-1)a(ax^2+bx+c)^{n-1}} + \frac{(m-1)c}{(2n-m-1)a}\int \frac{x^{m-2}\,dx}{(ax^2+bx+c)^n} - \frac{(n-m)b}{(2n-m-1)a}\int \frac{x^{m-1}\,dx}{(ax^2+bx+c)^n}$$

17.12.12 $$\int \frac{x^{2n-1}\,dx}{(ax^2+bx+c)^n} = \frac{1}{a}\int \frac{x^{2n-3}\,dx}{(ax^2+bx+c)^{n-1}} - \frac{c}{a}\int \frac{x^{2n-3}\,dx}{(ax^2+bx+c)^n} - \frac{b}{a}\int \frac{x^{2n-2}\,dx}{(ax^2+bx+c)^n}$$

17.12.13 $$\int \frac{dx}{x(ax^2+bx+c)^2} = \frac{1}{2c(ax^2+bx+c)} - \frac{b}{2c}\int \frac{dx}{(ax^2+bx+c)^2} + \frac{1}{c}\int \frac{dx}{x(ax^2+bx+c)}$$

17.12.14 $$\int \frac{dx}{x^2(ax^2+bx+c)^2} = -\frac{1}{cx(ax^2+bx+c)} - \frac{3a}{c}\int \frac{dx}{(ax^2+bx+c)^2} - \frac{2b}{c}\int \frac{dx}{x(ax^2+bx+c)^2}$$

17.12.15 $$\int \frac{dx}{x^m(ax^2+bx+c)^n} = -\frac{1}{(m-1)cx^{m-1}(ax^2+bx+c)^{n-1}} - \frac{(m+2n-3)a}{(m-1)c}\int \frac{dx}{x^{m-2}(ax^2+bx+c)^n} - \frac{(m+n-2)b}{(m-1)c}\int \frac{dx}{x^{m-1}(ax^2+bx+c)^n}$$

17.13 INTEGRALS INVOLVING $\sqrt{ax^2+bx+c}$

In the following results if $b^2 = 4ac$, $\sqrt{ax^2+bx+c} = \sqrt{a}(x+b/2a)$ and the results 17.1 can be used. If $b = 0$ use the results 17.9. If $a = 0$ or $c = 0$ use the results 17.2 and 17.5.

17.13.1 $$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \begin{cases} \dfrac{1}{\sqrt{a}}\ln\left(2\sqrt{a}\sqrt{ax^2+bx+c}+2ax+b\right) \\ -\dfrac{1}{\sqrt{-a}}\sin^{-1}\left(\dfrac{2ax+b}{\sqrt{b^2-4ac}}\right) \quad \text{or} \quad \dfrac{1}{\sqrt{a}}\sinh^{-1}\left(\dfrac{2ax+b}{\sqrt{4ac-b^2}}\right) \end{cases}$$

17.13.2 $$\int \frac{x\,dx}{\sqrt{ax^2+bx+c}} = \frac{\sqrt{ax^2+bx+c}}{a} - \frac{b}{2a}\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

17.13.3 $$\int \frac{x^2\,dx}{\sqrt{ax^2+bx+c}} = \frac{2ax-3b}{4a^2}\sqrt{ax^2+bx+c} + \frac{3b^2-4ac}{8a^2}\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

17.13.4 $$\int \frac{dx}{x\sqrt{ax^2+bx+c}} = \begin{cases} -\dfrac{1}{\sqrt{c}}\ln\left(\dfrac{2\sqrt{c}\sqrt{ax^2+bx+c}+bx+2c}{x}\right) \\ \dfrac{1}{\sqrt{-c}}\sin^{-1}\left(\dfrac{bx+2c}{|x|\sqrt{b^2-4ac}}\right) \quad \text{or} \quad -\dfrac{1}{\sqrt{c}}\sinh^{-1}\left(\dfrac{bx+2c}{|x|\sqrt{4ac-b^2}}\right) \end{cases}$$

17.13.5 $$\int \frac{dx}{x^2\sqrt{ax^2+bx+c}} = -\frac{\sqrt{ax^2+bx+c}}{cx} - \frac{b}{2c}\int \frac{dx}{x\sqrt{ax^2+bx+c}}$$

17.13.6 $$\int \sqrt{ax^2+bx+c}\,dx = \frac{(2ax+b)\sqrt{ax^2+bx+c}}{4a} + \frac{4ac-b^2}{8a}\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

17.13.7 $$\int x\sqrt{ax^2+bx+c}\,dx = \frac{(ax^2+bx+c)^{3/2}}{3a} - \frac{b(2ax+b)}{8a^2}\sqrt{ax^2+bx+c} - \frac{b(4ac-b^2)}{16a^2}\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

17.13.8 $$\int x^2\sqrt{ax^2+bx+c}\,dx = \frac{6ax-5b}{24a^2}(ax^2+bx+c)^{3/2} + \frac{5b^2-4ac}{16a^2}\int \sqrt{ax^2+bx+c}\,dx$$

17.13.9 $$\int \frac{\sqrt{ax^2+bx+c}}{x}\,dx = \sqrt{ax^2+bx+c} + \frac{b}{2}\int \frac{dx}{\sqrt{ax^2+bx+c}} + c\int \frac{dx}{x\sqrt{ax^2+bx+c}}$$

17.13.10 $$\int \frac{\sqrt{ax^2+bx+c}}{x^2}\,dx = -\frac{\sqrt{ax^2+bx+c}}{x} + a\int \frac{dx}{\sqrt{ax^2+bx+c}} + \frac{b}{2}\int \frac{dx}{x\sqrt{ax^2+bx+c}}$$

17.13.11 $$\int \frac{dx}{(ax^2+bx+c)^{3/2}} = \frac{2(2ax+b)}{(4ac-b^2)\sqrt{ax^2+bx+c}}$$

17.13.12 $$\int \frac{x\,dx}{(ax^2+bx+c)^{3/2}} = \frac{2(bx+2c)}{(b^2-4ac)\sqrt{ax^2+bx+c}}$$

17.13.13 $$\int \frac{x^2\,dx}{(ax^2+bx+c)^{3/2}} = \frac{(2b^2-4ac)x+2bc}{a(4ac-b^2)\sqrt{ax^2+bx+c}} + \frac{1}{a}\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

17.13.14 $$\int \frac{dx}{x(ax^2+bx+c)^{3/2}} = \frac{1}{c\sqrt{ax^2+bx+c}} + \frac{1}{c}\int \frac{dx}{x\sqrt{ax^2+bx+c}} - \frac{b}{2c}\int \frac{dx}{(ax^2+bx+c)^{3/2}}$$

17.13.15 $$\int \frac{dx}{x^2(ax^2+bx+c)^{3/2}} = -\frac{ax^2+2bx+c}{c^2x\sqrt{ax^2+bx+c}} + \frac{b^2-2ac}{2c^2}\int \frac{dx}{(ax^2+bx+c)^{3/2}} - \frac{3b}{2c^2}\int \frac{dx}{x\sqrt{ax^2+bx+c}}$$

17.13.16 $$\int (ax^2+bx+c)^{n+1/2}\,dx = \frac{(2ax+b)(ax^2+bx+c)^{n+1/2}}{4a(n+1)} + \frac{(2n+1)(4ac-b^2)}{8a(n+1)}\int (ax^2+bx+c)^{n-1/2}\,dx$$

17.13.17 $$\int x(ax^2+bx+c)^{n+1/2}\,dx = \frac{(ax^2+bx+c)^{n+3/2}}{a(2n+3)} - \frac{b}{2a}\int (ax^2+bx+c)^{n+1/2}\,dx$$

17.13.18 $$\int \frac{dx}{(ax^2+bx+c)^{n+1/2}} = \frac{2(2ax+b)}{(2n-1)(4ac-b^2)(ax^2+bx+c)^{n-1/2}} + \frac{8a(n-1)}{(2n-1)(4ac-b^2)}\int \frac{dx}{(ax^2+bx+c)^{n-1/2}}$$

17.13.19 $$\int \frac{dx}{x(ax^2+bx+c)^{n+1/2}} = \frac{1}{(2n-1)c(ax^2+bx+c)^{n-1/2}} + \frac{1}{c}\int \frac{dx}{x(ax^2+bx+c)^{n-1/2}} - \frac{b}{2c}\int \frac{dx}{(ax^2+bx+c)^{n+1/2}}$$

(14) INTEGRALS INVOLVING $x^3 + a^3$

Note that for formulas involving $x^3 - a^3$ replace a by $-a$.

17.14.1 $$\int \frac{dx}{x^3+a^3} = \frac{1}{6a^2}\ln\frac{(x+a)^2}{x^2-ax+a^2} + \frac{1}{a^2\sqrt{3}}\tan^{-1}\frac{2x-a}{a\sqrt{3}}$$

17.14.2 $$\int \frac{x\,dx}{x^3+a^3} = \frac{1}{6a}\ln\frac{x^2-ax+a^2}{(x+a)^2} + \frac{1}{a\sqrt{3}}\tan^{-1}\frac{2x-a}{a\sqrt{3}}$$

17.14.3 $$\int \frac{x^2\,dx}{x^3+a^3} = \frac{1}{3}\ln(x^3+a^3)$$

17.14.4 $$\int \frac{dx}{x(x^3+a^3)} = \frac{1}{3a^3}\ln\left(\frac{x^3}{x^3+a^3}\right)$$

17.14.5 $$\int \frac{dx}{x^2(x^3+a^3)} = -\frac{1}{a^3x} - \frac{1}{6a^4}\ln\frac{x^2-ax+a^2}{(x+a)^2} - \frac{1}{a^4\sqrt{3}}\tan^{-1}\frac{2x-a}{a\sqrt{3}}$$

17.14.6 $$\int \frac{dx}{(x^3+a^3)^2} = \frac{x}{3a^3(x^3+a^3)} + \frac{1}{9a^5}\ln\frac{(x+a)^2}{x^2-ax+a^2} + \frac{2}{3a^5\sqrt{3}}\tan^{-1}\frac{2x-a}{a\sqrt{3}}$$

17.14.7 $$\int \frac{x\,dx}{(x^3+a^3)^2} = \frac{x^2}{3a^3(x^3+a^3)} + \frac{1}{18a^4}\ln\frac{x^2-ax+a^2}{(x+a)^2} + \frac{1}{3a^4\sqrt{3}}\tan^{-1}\frac{2x-a}{a\sqrt{3}}$$

17.14.8 $$\int \frac{x^2\,dx}{(x^3+a^3)^2} = -\frac{1}{3(x^3+a^3)}$$

17.14.9 $$\int \frac{dx}{x(x^3+a^3)^2} = \frac{1}{3a^3(x^3+a^3)} + \frac{1}{3a^6}\ln\left(\frac{x^3}{x^3+a^3}\right)$$

17.14.10 $$\int \frac{dx}{x^2(x^3+a^3)^2} = -\frac{1}{a^6x} - \frac{x^2}{3a^6(x^3+a^3)} - \frac{4}{3a^6}\int\frac{x\,dx}{x^3+a^3}$$ [See 17.14.2]

17.14.11 $$\int \frac{x^m\,dx}{x^3+a^3} = \frac{x^{m-2}}{m-2} - a^3\int\frac{x^{m-3}\,dx}{x^3+a^3}$$

17.14.12 $$\int \frac{dx}{x^n(x^3+a^3)} = \frac{-1}{a^3(n-1)x^{n-1}} - \frac{1}{a^3}\int\frac{dx}{x^{n-3}(x^3+a^3)}$$

(15) INTEGRALS INVOLVING $x^4 \pm a^4$

17.15.1 $$\int \frac{dx}{x^4+a^4} = \frac{1}{4a^3\sqrt{2}}\ln\left(\frac{x^2+ax\sqrt{2}+a^2}{x^2-ax\sqrt{2}+a^2}\right) - \frac{1}{2a^3\sqrt{2}}\left[\tan^{-1}\left(1-\frac{x\sqrt{2}}{a}\right) - \tan^{-1}\left(1+\frac{x\sqrt{2}}{a}\right)\right]$$

17.15.2 $$\int \frac{x\,dx}{x^4+a^4} = \frac{1}{2a^2}\tan^{-1}\frac{x^2}{a^2}$$

17.15.3 $$\int \frac{x^2\,dx}{x^4+a^4} = \frac{1}{4a\sqrt{2}}\ln\left(\frac{x^2-ax\sqrt{2}+a^2}{x^2+ax\sqrt{2}+a^2}\right) - \frac{1}{2a\sqrt{2}}\left[\tan^{-1}\left(1-\frac{x\sqrt{2}}{a}\right) - \tan^{-1}\left(1+\frac{x\sqrt{2}}{a}\right)\right]$$

17.15.4 $$\int \frac{x^3\,dx}{x^4+a^4} = \frac{1}{4}\ln(x^4+a^4)$$

17.15.5 $$\int \frac{dx}{x(x^4+a^4)} = \frac{1}{4a^4}\ln\left(\frac{x^4}{x^4+a^4}\right)$$

17.15.6 $$\int \frac{dx}{x^2(x^4+a^4)} = -\frac{1}{a^4x} - \frac{1}{4a^5\sqrt{2}}\ln\left(\frac{x^2-ax\sqrt{2}+a^2}{x^2+ax\sqrt{2}+a^2}\right) + \frac{1}{2a^5\sqrt{2}}\left[\tan^{-1}\left(1-\frac{x\sqrt{2}}{a}\right) - \tan^{-1}\left(1+\frac{x\sqrt{2}}{a}\right)\right]$$

17.15.7 $$\int \frac{dx}{x^3(x^4+a^4)} = -\frac{1}{2a^4x^2} - \frac{1}{2a^6}\tan^{-1}\frac{x^2}{a^2}$$

17.15.8 $$\int \frac{dx}{x^4-a^4} = \frac{1}{4a^3}\ln\left(\frac{x-a}{x+a}\right) - \frac{1}{2a^3}\tan^{-1}\frac{x}{a}$$

17.15.9 $$\int \frac{x\,dx}{x^4-a^4} = \frac{1}{4a^2}\ln\left(\frac{x^2-a^2}{x^2+a^2}\right)$$

17.15.10 $$\int \frac{x^2\,dx}{x^4-a^4} = \frac{1}{4a}\ln\left(\frac{x-a}{x+a}\right) + \frac{1}{2a}\tan^{-1}\frac{x}{a}$$

17.15.11 $$\int \frac{x^3\,dx}{x^4-a^4} = \frac{1}{4}\ln(x^4-a^4)$$

17.15.12 $$\int \frac{dx}{x(x^4-a^4)} = \frac{1}{4a^4}\ln\left(\frac{x^4-a^4}{x^4}\right)$$

17.15.13 $$\int \frac{dx}{x^3(x^4-a^4)} = \frac{1}{a^4x} + \frac{1}{4a^5}\ln\left(\frac{x-a}{x+a}\right) + \frac{1}{2a^5}\tan^{-1}\frac{x}{a}$$

17.15.14 $$\int \frac{dx}{x^3(x^4-a^4)} = \frac{1}{2a^4x^2} + \frac{1}{4a^6}\ln\left(\frac{x^2-a^2}{x^2+a^2}\right)$$

(16) INTEGRALS INVOLVING $x^n \pm a^n$

17.16.1 $$\int \frac{dx}{x(x^n+a^n)} = \frac{1}{na^n}\ln\frac{x^n}{x^n+a^n}$$

17.16.2 $$\int \frac{x^{n-1}\,dx}{x^n+a^n} = \frac{1}{n}\ln(x^n+a^n)$$

17.16.3 $$\int \frac{x^m\,dx}{(x^n+a^n)^r} = \int \frac{x^{m-n}\,dx}{(x^n+a^n)^{r-1}} - a^n\int \frac{x^{m-n}\,dx}{(x^n+a^n)^r}$$

17.16.4 $$\int \frac{dx}{x^m(x^n+a^n)^r} = \frac{1}{a^n}\int \frac{dx}{x^m(x^n+a^n)^{r-1}} - \frac{1}{a^n}\int \frac{dx}{x^{m-n}(x^n+a^n)^r}$$

17.16.5 $$\int \frac{dx}{x\sqrt{x^n+a^n}} = \frac{1}{n\sqrt{a^n}}\ln\left(\frac{\sqrt{x^n+a^n}-\sqrt{a^n}}{\sqrt{x^n+a^n}+\sqrt{a^n}}\right)$$

17.16.6 $$\int \frac{dx}{x(x^n-a^n)} = \frac{1}{na^n}\ln\left(\frac{x^n-a^n}{x^n}\right)$$

17.16.7 $\int \frac{x^{n-1}\,dx}{x^n - a^n} = \frac{1}{n}\ln(x^n - a^n)$

17.16.8 $\int \frac{x^m\,dx}{(x^n - a^n)^r} = a^n \int \frac{x^{m-n}\,dx}{(x^n - a^n)^r} + \int \frac{x^{m-n}\,dx}{(x^n - a^n)^{r-1}}$

17.16.9 $\int \frac{dx}{x^m(x^n - a^n)^r} = \frac{1}{a^n}\int \frac{dx}{x^{m-n}(x^n - a^n)^r} - \frac{1}{a^n}\int \frac{dx}{x^m(x^n - a^n)^{r-1}}$

17.16.10 $\int \frac{dx}{x\sqrt{x^n - a^n}} = \frac{2}{n\sqrt{a^n}}\cos^{-1}\sqrt{\frac{a^n}{x^n}}$

17.16.11 $$\int \frac{x^{p-1}\,dx}{x^{2m} + a^{2m}} = \frac{1}{ma^{2m-p}}\sum_{k=1}^{m} \sin\frac{(2k-1)p\pi}{2m}\tan^{-1}\left(\frac{x + a\cos[(2k-1)\pi/2m]}{a\sin[(2k-1)\pi/2m]}\right)$$
$$-\frac{1}{2ma^{2m-p}}\sum_{k=1}^{m}\cos\frac{(2k-1)p\pi}{2m}\ln\left(x^2 + 2ax\cos\frac{(2k-1)\pi}{2m} + a^2\right)$$
where $0 < p \leqq 2m$.

17.16.12 $$\int \frac{x^{p-1}\,dx}{x^{2m} - a^{2m}} = \frac{1}{2ma^{2m-p}}\sum_{k=1}^{m-1}\cos\frac{kp\pi}{m}\ln\left(x^2 - 2ax\cos\frac{k\pi}{m} + a^2\right)$$
$$-\frac{1}{ma^{2m-p}}\sum_{k=1}^{m-1}\sin\frac{kp\pi}{m}\tan^{-1}\left(\frac{x - a\cos(k\pi/m)}{a\sin(k\pi/m)}\right)$$
$$+\frac{1}{2ma^{2m-p}}\{\ln(x-a) + (-1)^p\ln(x+a)\}$$
where $0 < p \leqq 2m$.

17.16.13 $$\int \frac{x^{p-1}\,dx}{x^{2m+1} + a^{2m+1}} = \frac{2(-1)^{p-1}}{(2m+1)a^{2m-p+1}}\sum_{k=1}^{m}\sin\frac{2kp\pi}{2m+1}\tan^{-1}\left(\frac{x + a\cos[2k\pi/(2m+1)]}{a\sin[2k\pi/(2m+1)]}\right)$$
$$-\frac{(-1)^{p-1}}{(2m+1)a^{2m-p+1}}\sum_{k=1}^{m}\cos\frac{2kp\pi}{2m+1}\ln\left(x^2 + 2ax\cos\frac{2k\pi}{2m+1} + a^2\right)$$
$$+\frac{(-1)^{p-1}\ln(x+a)}{(2m+1)a^{2m-p+1}}$$
where $0 < p \leqq 2m+1$.

17.16.14 $$\int \frac{x^{p-1}\,dx}{x^{2m+1} - a^{2m+1}} = \frac{-2}{(2m+1)a^{2m-p+1}}\sum_{k=1}^{m}\sin\frac{2kp\pi}{2m+1}\tan^{-1}\left(\frac{x - a\cos[2k\pi/(2m+1)]}{a\sin[2k\pi/(2m+1)]}\right)$$
$$+\frac{1}{(2m+1)a^{2m-p+1}}\sum_{k=1}^{m}\cos\frac{2kp\pi}{2m+1}\ln\left(x^2 - 2ax\cos\frac{2k\pi}{2m+1} + a^2\right)$$
$$+\frac{\ln(x-a)}{(2m+1)a^{2m-p+1}}$$
where $0 < p \leqq 2m+1$.

(17) INTEGRALS INVOLVING $\sin ax$

17.17.1 $\int \sin ax\,dx = -\frac{\cos ax}{a}$

17.17.2 $\int x\sin ax\,dx = \frac{\sin ax}{a^2} - \frac{x\cos ax}{a}$

17.17.3 $$\int x^2 \sin ax\,dx = \frac{2x}{a^2}\sin ax + \left(\frac{2}{a^3} - \frac{x^2}{a}\right)\cos ax$$

17.17.4 $$\int x^3 \sin ax\,dx = \left(\frac{3x^2}{a^2} - \frac{6}{a^4}\right)\sin ax + \left(\frac{6x}{a^3} - \frac{x^3}{a}\right)\cos ax$$

17.17.5 $$\int \frac{\sin ax}{x}\,dx = ax - \frac{(ax)^3}{3\cdot 3!} + \frac{(ax)^5}{5\cdot 5!} - \cdots$$

17.17.6 $$\int \frac{\sin ax}{x^2}\,dx = -\frac{\sin ax}{x} + a\int \frac{\cos ax}{x}\,dx$$ [see 17.18.5]

17.17.7 $$\int \frac{dx}{\sin ax} = \frac{1}{a}\ln(\csc ax - \cot ax) = \frac{1}{a}\ln\tan\frac{ax}{2}$$

17.17.8 $$\int \frac{x\,dx}{\sin ax} = \frac{1}{a^2}\left\{ax + \frac{(ax)^3}{18} + \frac{7(ax)^5}{1800} + \cdots + \frac{2(2^{2n-1}-1)B_n(ax)^{2n+1}}{(2n+1)!} + \cdots\right\}$$

17.17.9 $$\int \sin^2 ax\,dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$$

17.17.10 $$\int x\sin^2 ax\,dx = \frac{x^2}{4} - \frac{x\sin 2ax}{4a} - \frac{\cos 2ax}{8a^2}$$

17.17.11 $$\int \sin^3 ax\,dx = -\frac{\cos ax}{a} + \frac{\cos^3 ax}{3a}$$

17.17.12 $$\int \sin^4 ax\,dx = \frac{3x}{8} - \frac{\sin 2ax}{4a} + \frac{\sin 4ax}{32a}$$

17.17.13 $$\int \frac{dx}{\sin^2 ax} = -\frac{1}{a}\cot ax$$

17.17.14 $$\int \frac{dx}{\sin^3 ax} = -\frac{\cos ax}{2a\sin^2 ax} + \frac{1}{2a}\ln\tan\frac{ax}{2}$$

17.17.15 $$\int \sin px\sin qx\,dx = \frac{\sin(p-q)x}{2(p-q)} - \frac{\sin(p+q)x}{2(p+q)}$$ [If $p = \pm q$, see 17.17.9.]

17.17.16 $$\int \frac{dx}{1-\sin ax} = \frac{1}{a}\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$$

17.17.17 $$\int \frac{x\,dx}{1-\sin ax} = \frac{x}{a}\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right) + \frac{2}{a^2}\ln\sin\left(\frac{\pi}{4} - \frac{ax}{2}\right)$$

17.17.18 $$\int \frac{dx}{1+\sin ax} = -\frac{1}{a}\tan\left(\frac{\pi}{4} - \frac{ax}{2}\right)$$

17.17.19 $$\int \frac{x\,dx}{1+\sin ax} = -\frac{x}{a}\tan\left(\frac{\pi}{4} - \frac{ax}{2}\right) + \frac{2}{a^2}\ln\sin\left(\frac{\pi}{4} + \frac{ax}{2}\right)$$

17.17.20 $$\int \frac{dx}{(1-\sin ax)^2} = \frac{1}{2a}\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right) + \frac{1}{6a}\tan^3\left(\frac{\pi}{4} + \frac{ax}{2}\right)$$

17.17.21 $$\int \frac{dx}{(1+\sin ax)^2} = -\frac{1}{2a}\tan\left(\frac{\pi}{4} - \frac{ax}{2}\right) - \frac{1}{6a}\tan^3\left(\frac{\pi}{4} - \frac{ax}{2}\right)$$

17.17.22 $$\int \frac{dx}{p+q\sin ax} = \begin{cases} \dfrac{2}{a\sqrt{p^2-q^2}}\tan^{-1}\dfrac{p\tan\frac{1}{2}ax+q}{\sqrt{p^2-q^2}} \\ \dfrac{1}{a\sqrt{q^2-p^2}}\ln\left(\dfrac{p\tan\frac{1}{2}ax+q-\sqrt{q^2-p^2}}{p\tan\frac{1}{2}ax+q+\sqrt{q^2-p^2}}\right) \end{cases}$$

If $p = \pm q$, see 17.17.16 and 17.17.18.

17.17.23 $$\int \frac{dx}{(p+q\sin ax)^2} = \frac{q\cos ax}{a(p^2-q^2)(p+q\sin ax)} + \frac{p}{p^2-q^2}\int\frac{dx}{p+q\sin ax}$$

If $p = \pm q$, see 17.17.20 and 17.17.21.

17.17.24 $$\int \frac{dx}{p^2+q^2\sin^2 ax} = \frac{1}{ap\sqrt{p^2+q^2}}\tan^{-1}\frac{\sqrt{p^2+q^2}\tan ax}{p}$$

17.17.25 $$\int \frac{dx}{p^2-q^2\sin^2 ax} = \begin{cases} \dfrac{1}{ap\sqrt{p^2-q^2}}\tan^{-1}\dfrac{\sqrt{p^2-q^2}\tan ax}{p} \\ \dfrac{1}{2ap\sqrt{q^2-p^2}}\ln\left(\dfrac{\sqrt{q^2-p^2}\tan ax+p}{\sqrt{q^2-p^2}\tan ax-p}\right) \end{cases}$$

17.17.26 $$\int x^m\sin ax\,dx = -\frac{x^m\cos ax}{a} + \frac{mx^{m-1}\sin ax}{a^2} - \frac{m(m-1)}{a^2}\int x^{m-2}\sin ax\,dx$$

17.17.27 $$\int \frac{\sin ax}{x^n}dx = -\frac{\sin ax}{(n-1)x^{n-1}} + \frac{a}{n-1}\int\frac{\cos ax}{x^{n-1}}dx$$ [see 17.18.30]

17.17.28 $$\int \sin^n ax\,dx = -\frac{\sin^{n-1}ax\cos ax}{an} + \frac{n-1}{n}\int\sin^{n-2}ax\,dx$$

17.17.29 $$\int \frac{dx}{\sin^n ax} = \frac{-\cos ax}{a(n-1)\sin^{n-1}ax} + \frac{n-2}{n-1}\int\frac{dx}{\sin^{n-2}ax}$$

17.17.30 $$\int \frac{x\,dx}{\sin^n ax} = \frac{-x\cos ax}{a(n-1)\sin^{n-1}ax} - \frac{1}{a^2(n-1)(n-2)\sin^{n-2}ax} + \frac{n-2}{n-1}\int\frac{x\,dx}{\sin^{n-2}ax}$$

(18) INTEGRALS INVOLVING $\cos ax$

17.18.1 $$\int \cos ax\,dx = \frac{\sin ax}{a}$$

17.18.2 $$\int x\cos ax\,dx = \frac{\cos ax}{a^2} + \frac{x\sin ax}{a}$$

17.18.3 $$\int x^2\cos ax\,dx = \frac{2x}{a^2}\cos ax + \left(\frac{x^2}{a} - \frac{2}{a^3}\right)\sin ax$$

17.18.4 $$\int x^3\cos ax\,dx = \left(\frac{3x^2}{a^2} - \frac{6}{a^4}\right)\cos ax + \left(\frac{x^3}{a} - \frac{6x}{a^3}\right)\sin ax$$

17.18.5 $$\int \frac{\cos ax}{x}dx = \ln x - \frac{(ax)^2}{2\cdot 2!} + \frac{(ax)^4}{4\cdot 4!} - \frac{(ax)^6}{6\cdot 6!} + \cdots$$

17.18.6 $$\int \frac{\cos ax}{x^2}dx = -\frac{\cos ax}{x} - a\int\frac{\sin ax}{x}dx$$ [See 17.17.5]

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17.18.7 $$\int \frac{dx}{\cos ax} = \frac{1}{a}\ln(\sec ax + \tan ax) = \frac{1}{a}\ln\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)$$

17.18.8 $$\int \frac{x\,dx}{\cos ax} = \frac{1}{a^2}\left\{\frac{(ax)^2}{2}+\frac{(ax)^4}{8}+\frac{5(ax)^6}{144}+\cdots+\frac{E_n(ax)^{2n+2}}{(2n+2)(2n)!}+\cdots\right\}$$

17.18.9 $$\int \cos^2 ax\,dx = \frac{x}{2}+\frac{\sin 2ax}{4a}$$

17.18.10 $$\int x\cos^2 ax\,dx = \frac{x^2}{4}+\frac{x\sin 2ax}{4a}+\frac{\cos 2ax}{8a^2}$$

17.18.11 $$\int \cos^3 ax\,dx = \frac{\sin ax}{a}-\frac{\sin^3 ax}{3a}$$

17.18.12 $$\int \cos^4 ax\,dx = \frac{3x}{8}+\frac{\sin 2ax}{4a}+\frac{\sin 4ax}{32a}$$

17.18.13 $$\int \frac{dx}{\cos^2 ax} = \frac{\tan ax}{a}$$

17.18.14 $$\int \frac{dx}{\cos^3 ax} = \frac{\sin ax}{2a\cos^2 ax}+\frac{1}{2a}\ln\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)$$

17.18.15 $$\int \cos ax\cos px\,dx = \frac{\sin(a-p)x}{2(a-p)}+\frac{\sin(a+p)x}{2(a+p)}$$ [If $a = \pm p$, see 17.18.9.]

17.18.16 $$\int \frac{dx}{1-\cos ax} = -\frac{1}{a}\cot\frac{ax}{2}$$

17.18.17 $$\int \frac{x\,dx}{1-\cos ax} = -\frac{x}{a}\cot\frac{ax}{2}+\frac{2}{a^2}\ln\sin\frac{ax}{2}$$

17.18.18 $$\int \frac{dx}{1+\cos ax} = \frac{1}{a}\tan\frac{ax}{2}$$

17.18.19 $$\int \frac{x\,dx}{1+\cos ax} = \frac{x}{a}\tan\frac{ax}{2}+\frac{2}{a^2}\ln\cos\frac{ax}{2}$$

17.18.20 $$\int \frac{dx}{(1-\cos ax)^2} = -\frac{1}{2a}\cot\frac{ax}{2}-\frac{1}{6a}\cot^3\frac{ax}{2}$$

17.18.21 $$\int \frac{dx}{(1+\cos ax)^2} = \frac{1}{2a}\tan\frac{ax}{2}+\frac{1}{6a}\tan^3\frac{ax}{2}$$

17.18.22 $$\int \frac{dx}{p+q\cos ax} = \begin{cases} \dfrac{2}{a\sqrt{p^2-q^2}}\tan^{-1}\sqrt{(p-q)/(p+q)}\tan\tfrac{1}{2}ax \\ \dfrac{1}{a\sqrt{q^2-p^2}}\ln\left(\dfrac{\tan\frac{1}{2}ax+\sqrt{(q+p)/(q-p)}}{\tan\frac{1}{2}ax-\sqrt{(q+p)/(q-p)}}\right)\end{cases}$$ [If $p = \pm q$, see 17.18.16 and 17.18.18.]

17.18.23 $$\int \frac{dx}{(p+q\cos ax)^2} = \frac{q\sin ax}{a(q^2-p^2)(p+q\cos ax)}-\frac{p}{q^2-p^2}\int\frac{dx}{p+q\cos ax}$$ [If $p = \pm q$ see 17.18.19 and 17.18.20.]

17.18.24 $$\int \frac{dx}{p^2+q^2\cos^2 ax} = \frac{1}{ap\sqrt{p^2+q^2}}\tan^{-1}\frac{p\tan ax}{\sqrt{p^2+q^2}}$$

17.18.25 $\displaystyle\int \frac{dx}{p^2 - q^2\cos^2 ax} = \begin{cases} \dfrac{1}{ap\sqrt{p^2-q^2}}\tan^{-1}\dfrac{p\tan ax}{\sqrt{p^2-q^2}} \\ \dfrac{1}{2ap\sqrt{q^2-p^2}}\ln\left(\dfrac{p\tan ax - \sqrt{q^2-p^2}}{p\tan ax + \sqrt{q^2-p^2}}\right) \end{cases}$

17.18.26 $\displaystyle\int x^m \cos ax\,dx = \frac{x^m \sin ax}{a} + \frac{mx^{m-1}}{a^2}\cos ax - \frac{m(m-1)}{a^2}\int x^{m-2}\cos ax\,dx$

17.18.27 $\displaystyle\int \frac{\cos ax}{x^n}dx = -\frac{\cos ax}{(n-1)x^{n-1}} - \frac{a}{n-1}\int \frac{\sin ax}{x^{n-1}}dx$ [See 17.17.27]

17.18.28 $\displaystyle\int \cos^n ax\,dx = \frac{\sin ax \cos^{n-1} ax}{an} + \frac{n-1}{n}\int \cos^{n-2} ax\,dx$

17.18.29 $\displaystyle\int \frac{dx}{\cos^n ax} = \frac{\sin ax}{a(n-1)\cos^{n-1} ax} + \frac{n-2}{n-1}\int \frac{dx}{\cos^{n-2} ax}$

17.18.30 $\displaystyle\int \frac{x\,dx}{\cos^n ax} = \frac{x\sin ax}{a(n-1)\cos^{n-1} ax} - \frac{1}{a^2(n-1)(n-2)\cos^{n-2} ax} + \frac{n-2}{n-1}\int \frac{x\,dx}{\cos^{n-2} ax}$

INTEGRALS INVOLVING $\sin ax$ AND $\cos ax$

17.19.1 $\displaystyle\int \sin ax \cos ax\,dx = \frac{\sin^2 ax}{2a}$

17.19.2 $\displaystyle\int \sin px \cos qx\,dx = -\frac{\cos(p-q)x}{2(p-q)} - \frac{\cos(p+q)x}{2(p+q)}$

17.19.3 $\displaystyle\int \sin^n ax \cos ax\,dx = \frac{\sin^{n+1} ax}{(n+1)a}$ [If $n = -1$, see 17.21.1.]

17.19.4 $\displaystyle\int \cos^n ax \sin ax\,dx = -\frac{\cos^{n+1} ax}{(n+1)a}$ [If $n = -1$, see 17.20.1.]

17.19.5 $\displaystyle\int \sin^2 ax \cos^2 ax\,dx = \frac{x}{8} - \frac{\sin 4ax}{32a}$

17.19.6 $\displaystyle\int \frac{dx}{\sin ax \cos ax} = \frac{1}{a}\ln\tan ax$

17.19.7 $\displaystyle\int \frac{dx}{\sin^2 ax \cos ax} = \frac{1}{a}\ln\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right) - \frac{1}{a\sin ax}$

17.19.8 $\displaystyle\int \frac{dx}{\sin ax \cos^2 ax} = \frac{1}{a}\ln\tan\frac{ax}{2} + \frac{1}{a\cos ax}$

17.19.9 $\displaystyle\int \frac{dx}{\sin^2 ax \cos^2 ax} = -\frac{2\cot 2ax}{a}$

17.19.10 $\displaystyle\int \frac{\sin^2 ax}{\cos ax}dx = -\frac{\sin ax}{a} + \frac{1}{a}\ln\tan\left(\frac{ax}{2} + \frac{\pi}{4}\right)$

17.19.11 $$\int \frac{\cos^2 ax}{\sin ax}\,dx = \frac{\cos ax}{a} + \frac{1}{a}\ln\tan\frac{ax}{2}$$

17.19.12 $$\int \frac{dx}{\cos ax(1 \pm \sin ax)} = \mp \frac{1}{2a(1 \pm \sin ax)} + \frac{1}{2a}\ln\tan\left(\frac{ax}{2} + \frac{\pi}{4}\right)$$

17.19.13 $$\int \frac{dx}{\sin ax(1 \pm \cos ax)} = \pm \frac{1}{2a(1 \pm \cos ax)} + \frac{1}{2a}\ln\tan\frac{ax}{2}$$

17.19.14 $$\int \frac{dx}{\sin ax \pm \cos ax} = \frac{1}{a\sqrt{2}}\ln\tan\left(\frac{ax}{2} \pm \frac{\pi}{8}\right)$$

17.19.15 $$\int \frac{\sin ax\,dx}{\sin ax \pm \cos ax} = \frac{x}{2} \mp \frac{1}{2a}\ln(\sin ax \pm \cos ax)$$

17.19.16 $$\int \frac{\cos ax\,dx}{\sin ax \pm \cos ax} = \pm\frac{x}{2} + \frac{1}{2a}\ln(\sin ax \pm \cos ax)$$

17.19.17 $$\int \frac{\sin ax\,dx}{p + q\cos ax} = -\frac{1}{aq}\ln(p + q\cos ax)$$

17.19.18 $$\int \frac{\cos ax\,dx}{p + q\sin ax} = \frac{1}{aq}\ln(p + q\sin ax)$$

17.19.19 $$\int \frac{\sin ax\,dx}{(p + q\cos ax)^n} = \frac{1}{aq(n-1)(p + q\cos ax)^{n-1}}$$

17.19.20 $$\int \frac{\cos ax\,dx}{(p + q\sin ax)^n} = \frac{-1}{aq(n-1)(p + q\sin ax)^{n-1}}$$

17.19.21 $$\int \frac{dx}{p\sin ax + q\cos ax} = \frac{1}{a\sqrt{p^2+q^2}}\ln\tan\left(\frac{ax + \tan^{-1}(q/p)}{2}\right)$$

17.19.22 $$\int \frac{dx}{p\sin ax + q\cos ax + r} = \begin{cases} \dfrac{2}{a\sqrt{r^2-p^2-q^2}}\tan^{-1}\left(\dfrac{p + (r-q)\tan(ax/2)}{\sqrt{r^2-p^2-q^2}}\right) \\ \dfrac{1}{a\sqrt{p^2+q^2-r^2}}\ln\left(\dfrac{p - \sqrt{p^2+q^2-r^2} + (r-q)\tan(ax/2)}{p + \sqrt{p^2+q^2-r^2} + (r-q)\tan(ax/2)}\right) \end{cases}$$

If $r = q$ see 17.19.23. If $r^2 = p^2 + q^2$ see 17.19.24.

17.19.23 $$\int \frac{dx}{p\sin ax + q(1 + \cos ax)} = \frac{1}{ap}\ln\left(q + p\tan\frac{ax}{2}\right)$$

17.19.24 $$\int \frac{dx}{p\sin ax + q\cos ax \pm \sqrt{p^2+q^2}} = \frac{-1}{a\sqrt{p^2+q^2}}\tan\left(\frac{\pi}{4} \mp \frac{ax + \tan^{-1}(q/p)}{2}\right)$$

17.19.25 $$\int \frac{dx}{p^2\sin^2 ax + q^2\cos^2 ax} = \frac{1}{apq}\tan^{-1}\left(\frac{p\tan ax}{q}\right)$$

17.19.26 $$\int \frac{dx}{p^2\sin^2 ax - q^2\cos^2 ax} = \frac{1}{2apq}\ln\left(\frac{p\tan ax - q}{p\tan ax + q}\right)$$

17.19.27 $$\int \sin^m ax\cos^n ax\,dx = \begin{cases} -\dfrac{\sin^{m-1} ax\cos^{n+1} ax}{a(m+n)} + \dfrac{m-1}{m+n}\displaystyle\int \sin^{m-2} ax\cos^n ax\,dx \\ \dfrac{\sin^{m+1} ax\cos^{n-1} ax}{a(m+n)} + \dfrac{n-1}{m+n}\displaystyle\int \sin^m ax\cos^{n-2} ax\,dx \end{cases}$$

17.19.28 $$\int \frac{\sin^m ax}{\cos^n ax}dx = \begin{cases} \dfrac{\sin^{m-1} ax}{a(n-1)\cos^{n-1} ax} - \dfrac{m-1}{n-1}\displaystyle\int \frac{\sin^{m-2} ax}{\cos^{n-2} ax}dx \\ \dfrac{\sin^{m+1} ax}{a(n-1)\cos^{n-1} ax} - \dfrac{m-n+2}{n-1}\displaystyle\int \frac{\sin^{m} ax}{\cos^{n-2} ax}dx \\ \dfrac{-\sin^{m-1} ax}{a(m-n)\cos^{n-1} ax} + \dfrac{m-1}{m-n}\displaystyle\int \frac{\sin^{m-2} ax}{\cos^{n} ax}dx \end{cases}$$

17.19.29 $$\int \frac{\cos^m ax}{\sin^n ax}dx = \begin{cases} \dfrac{-\cos^{m-1} ax}{a(n-1)\sin^{n-1} ax} - \dfrac{m-1}{n-1}\displaystyle\int \frac{\cos^{m-2} ax}{\sin^{n-2} ax}dx \\ \dfrac{-\cos^{m+1} ax}{a(n-1)\sin^{n-1} ax} - \dfrac{m-n+2}{n-1}\displaystyle\int \frac{\cos^{m} ax}{\sin^{n-2} ax}dx \\ \dfrac{\cos^{m-1} ax}{a(m-n)\sin^{n-1} ax} + \dfrac{m-1}{m-n}\displaystyle\int \frac{\cos^{m-2} ax}{\sin^{n} ax}dx \end{cases}$$

17.19.30 $$\int \frac{dx}{\sin^m ax\cos^n ax} = \begin{cases} \dfrac{1}{a(n-1)\sin^{m-1} ax\cos^{n-1} ax} + \dfrac{m+n-2}{n-1}\displaystyle\int \frac{dx}{\sin^m ax\cos^{n-2} ax} \\ \dfrac{-1}{a(m-1)\sin^{m-1} ax\cos^{n-1} ax} + \dfrac{m+n-2}{m-1}\displaystyle\int \frac{dx}{\sin^{m-2} ax\cos^{n} ax} \end{cases}$$

INTEGRALS INVOLVING tan ax

17.20.1 $$\int \tan ax\,dx = -\frac{1}{a}\ln\cos ax = \frac{1}{a}\ln\sec ax$$

17.20.2 $$\int \tan^2 ax\,dx = \frac{\tan ax}{a} - x$$

17.20.3 $$\int \tan^3 ax\,dx = \frac{\tan^2 ax}{2a} + \frac{1}{a}\ln\cos ax$$

17.20.4 $$\int \tan^n ax\sec^2 ax\,dx = \frac{\tan^{n+1} ax}{(n+1)a}$$

17.20.5 $$\int \frac{\sec^2 ax}{\tan ax}dx = \frac{1}{a}\ln\tan ax$$

17.20.6 $$\int \frac{dx}{\tan ax} = \frac{1}{a}\ln\sin ax$$

17.20.7 $$\int x\tan ax\,dx = \frac{1}{a^2}\left\{\frac{(ax)^3}{3} + \frac{(ax)^5}{15} + \frac{2(ax)^7}{105} + \cdots + \frac{2^{2n}(2^{2n}-1)B_n(ax)^{2n+1}}{(2n+1)!} + \cdots\right\}$$

17.20.8 $$\int \frac{\tan ax}{x}dx = ax + \frac{(ax)^3}{9} + \frac{2(ax)^5}{75} + \cdots + \frac{2^{2n}(2^{2n}-1)B_n(ax)^{2n-1}}{(2n-1)(2n)!} + \cdots$$

17.20.9 $$\int x\tan^2 ax\,dx = \frac{x\tan ax}{a} + \frac{1}{a^2}\ln\cos ax - \frac{x^2}{2}$$

17.20.10 $\displaystyle\int \frac{dx}{p+q\tan ax} = \frac{px}{p^2+q^2} + \frac{q}{a(p^2+q^2)}\ln(q\sin ax + p\cos ax)$

17.20.11 $\displaystyle\int \tan^n ax\,dx = \frac{\tan^{n-1} ax}{(n-1)a} - \int \tan^{n-2} ax\,dx$

(21) INTEGRALS INVOLVING $\cot ax$

17.21.1 $\displaystyle\int \cot ax\,dx = \frac{1}{a}\ln\sin ax$

17.21.2 $\displaystyle\int \cot^2 ax\,dx = -\frac{\cot ax}{a} - x$

17.21.3 $\displaystyle\int \cot^3 ax\,dx = -\frac{\cot^2 ax}{2a} - \frac{1}{a}\ln\sin ax$

17.21.4 $\displaystyle\int \cot^n ax\csc^2 ax\,dx = -\frac{\cot^{n+1} ax}{(n+1)a}$

17.21.5 $\displaystyle\int \frac{\csc^2 ax}{\cot ax}\,dx = -\frac{1}{a}\ln\cot ax$

17.21.6 $\displaystyle\int \frac{dx}{\cot ax} = -\frac{1}{a}\ln\cos ax$

17.21.7 $\displaystyle\int x\cot ax\,dx = \frac{1}{a^2}\left\{ax - \frac{(ax)^3}{9} - \frac{(ax)^5}{225} - \cdots - \frac{2^{2n}B_n(ax)^{2n+1}}{(2n+1)!} - \cdots\right\}$

17.21.8 $\displaystyle\int \frac{\cot ax}{x}\,dx = -\frac{1}{ax} - \frac{ax}{3} - \frac{(ax)^3}{135} - \cdots - \frac{2^{2n}B_n(ax)^{2n-1}}{(2n-1)(2n)!} - \cdots$

17.21.9 $\displaystyle\int x\cot^2 ax\,dx = -\frac{x\cot ax}{a} + \frac{1}{a^2}\ln\sin ax - \frac{x^2}{2}$

17.21.10 $\displaystyle\int \frac{dx}{p+q\cot ax} = \frac{px}{p^2+q^2} - \frac{q}{a(p^2+q^2)}\ln(q\sin ax + q\cos ax)$

17.21.11 $\displaystyle\int \cot^n ax\,dx = -\frac{\cot^{n-1} ax}{(n-1)a} - \int \cot^{n-2} ax\,dx$

(22) INTEGRALS INVOLVING $\sec ax$

17.22.1 $\displaystyle\int \sec ax\,dx = \frac{1}{a}\ln(\sec ax + \tan ax) = \frac{1}{a}\ln\tan\left(\frac{ax}{2} + \frac{\pi}{4}\right)$

17.22.2 $\displaystyle\int \sec^2 ax\,dx = \frac{\tan ax}{a}$

17.22.3 $\displaystyle\int \sec^3 ax\,dx = \frac{\sec ax\tan ax}{2a} + \frac{1}{2a}\ln(\sec ax + \tan ax)$

17.22.4 $\int \sec^n ax \tan ax\, dx = \frac{\sec^n ax}{na}$

17.22.5 $\int \frac{dx}{\sec ax} = \frac{\sin ax}{a}$

17.22.6 $\int x \sec ax\, dx = \frac{1}{a^2}\left\{\frac{(ax)^2}{2} + \frac{(ax)^4}{8} + \frac{5(ax)^6}{144} + \cdots + \frac{E_n(ax)^{2n+2}}{(2n+2)(2n)!} + \cdots\right\}$

17.22.7 $\int \frac{\sec ax}{x} dx = \ln x + \frac{(ax)^2}{4} + \frac{5(ax)^4}{96} + \frac{61(ax)^6}{4320} + \cdots + \frac{E_n(ax)^{2n}}{2n(2n)!} + \cdots$

17.22.8 $\int x \sec^2 ax\, dx = \frac{x}{a}\tan ax + \frac{1}{a^2}\ln \cos ax$

17.22.9 $\int \frac{dx}{q + p \sec ax} = \frac{x}{q} - \frac{p}{q}\int \frac{dx}{p + q\cos ax}$

17.22.10 $\int \sec^n ax\, dx = \frac{\sec^{n-2} ax \tan ax}{a(n-1)} + \frac{n-2}{n-1}\int \sec^{n-2} ax\, dx$

(23) INTEGRALS INVOLVING $\csc ax$

17.23.1 $\int \csc ax\, dx = \frac{1}{a}\ln(\csc ax - \cot ax) = \frac{1}{a}\ln\tan\frac{ax}{2}$

17.23.2 $\int \csc^2 ax\, dx = -\frac{\cot ax}{a}$

17.23.3 $\int \csc^3 ax\, dx = -\frac{\csc ax \cot ax}{2a} + \frac{1}{2a}\ln\tan\frac{ax}{2}$

17.23.4 $\int \csc^n ax \cot ax\, dx = -\frac{\csc^n ax}{na}$

17.23.5 $\int \frac{dx}{\csc ax} = -\frac{\cos ax}{a}$

17.23.6 $\int x \csc ax\, dx = \frac{1}{a^2}\left\{ax + \frac{(ax)^3}{18} + \frac{7(ax)^5}{1800} + \cdots + \frac{2(2^{2n-1}-1)B_n(ax)^{2n+1}}{(2n+1)!} + \cdots\right\}$

17.23.7 $\int \frac{\csc ax}{x} dx = -\frac{1}{ax} + \frac{ax}{6} + \frac{7(ax)^3}{1080} + \cdots + \frac{2(2^{2n-1}-1)B_n(ax)^{2n-1}}{(2n-1)(2n)!} + \cdots$

17.23.8 $\int x \csc^2 ax\, dx = -\frac{x \cot ax}{a} + \frac{1}{a^2}\ln \sin ax$

17.23.9 $\int \frac{dx}{q + p\csc ax} = \frac{x}{q} - \frac{p}{q}\int \frac{dx}{p + q\sin ax}$ [See 17.17.22]

17.23.10 $\int \csc^n ax\, dx = -\frac{\csc^{n-2} ax \cot ax}{a(n-1)} + \frac{n-2}{n-1}\int \csc^{n-2} ax\, dx$

(24) INTEGRALS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

17.24.1 $$\int \sin^{-1}\frac{x}{a}\,dx = x\sin^{-1}\frac{x}{a} + \sqrt{a^2-x^2}$$

17.24.2 $$\int x\sin^{-1}\frac{x}{a}\,dx = \left(\frac{x^2}{2}-\frac{a^2}{4}\right)\sin^{-1}\frac{x}{a} + \frac{x\sqrt{a^2-x^2}}{4}$$

17.24.3 $$\int x^2\sin^{-1}\frac{x}{a}\,dx = \frac{x^3}{3}\sin^{-1}\frac{x}{a} + \frac{(x^2+2a^2)\sqrt{a^2-x^2}}{9}$$

17.24.4 $$\int \frac{\sin^{-1}(x/a)}{x}\,dx = \frac{x}{a} + \frac{(x/a)^3}{2\cdot 3\cdot 3} + \frac{1\cdot 3(x/a)^5}{2\cdot 4\cdot 5\cdot 5} + \frac{1\cdot 3\cdot 5(x/a)^7}{2\cdot 4\cdot 6\cdot 7\cdot 7} + \cdots$$

17.24.5 $$\int \frac{\sin^{-1}(x/a)}{x^2}\,dx = -\frac{\sin^{-1}(x/a)}{x} - \frac{1}{a}\ln\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)$$

17.24.6 $$\int \left(\sin^{-1}\frac{x}{a}\right)^2 dx = x\left(\sin^{-1}\frac{x}{a}\right)^2 - 2x + 2\sqrt{a^2-x^2}\sin^{-1}\frac{x}{a}$$

17.24.7 $$\int \cos^{-1}\frac{x}{a}\,dx = x\cos^{-1}\frac{x}{a} - \sqrt{a^2-x^2}$$

17.24.8 $$\int x\cos^{-1}\frac{x}{a}\,dx = \left(\frac{x^2}{2}-\frac{a^2}{4}\right)\cos^{-1}\frac{x}{a} - \frac{x\sqrt{a^2-x^2}}{4}$$

17.24.9 $$\int x^2\cos^{-1}\frac{x}{a}\,dx = \frac{x^3}{3}\cos^{-1}\frac{x}{a} - \frac{(x^2+2a^2)\sqrt{a^2-x^2}}{9}$$

17.24.10 $$\int \frac{\cos^{-1}(x/a)}{x}\,dx = \frac{\pi}{2}\ln x - \int \frac{\sin^{-1}(x/a)}{x}\,dx$$ [See 17.24.4]

17.24.11 $$\int \frac{\cos^{-1}(x/a)}{x^2}\,dx = -\frac{\cos^{-1}(x/a)}{x} + \frac{1}{a}\ln\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)$$

17.24.12 $$\int \left(\cos^{-1}\frac{x}{a}\right)^2 dx = x\left(\cos^{-1}\frac{x}{a}\right)^2 - 2x - 2\sqrt{a^2-x^2}\cos^{-1}\frac{x}{a}$$

17.24.13 $$\int \tan^{-1}\frac{x}{a}\,dx = x\tan^{-1}\frac{x}{a} - \frac{a}{2}\ln(x^2+a^2)$$

17.24.14 $$\int x\tan^{-1}\frac{x}{a}\,dx = \tfrac{1}{2}(x^2+a^2)\tan^{-1}\frac{x}{a} - \frac{ax}{2}$$

17.24.15 $$\int x^2\tan^{-1}\frac{x}{a}\,dx = \frac{x^3}{3}\tan^{-1}\frac{x}{a} - \frac{ax^2}{6} + \frac{a^3}{6}\ln(x^2+a^2)$$

17.24.16 $$\int \frac{\tan^{-1}(x/a)}{x}\,dx = \frac{x}{a} - \frac{(x/a)^3}{3^2} + \frac{(x/a)^5}{5^2} - \frac{(x/a)^7}{7^2} + \cdots$$

17.24.17 $$\int \frac{\tan^{-1}(x/a)}{x^2}\,dx = -\frac{1}{x}\tan^{-1}\frac{x}{a} - \frac{1}{2a}\ln\left(\frac{x^2+a^2}{x^2}\right)$$

17.24.18 $$\int \cot^{-1}\frac{x}{a}\,dx = x\cot^{-1}\frac{x}{a} + \frac{a}{2}\ln(x^2+a^2)$$

17.24.19 $$\int x\cot^{-1}\frac{x}{a}\,dx = \tfrac{1}{2}(x^2+a^2)\cot^{-1}\frac{x}{a} + \frac{ax}{2}$$

17.24.20 $$\int x^2\cot^{-1}\frac{x}{a}\,dx = \frac{x^3}{3}\cot^{-1}\frac{x}{a} + \frac{ax^2}{6} - \frac{a^3}{6}\ln(x^2+a^2)$$

17.24.21 $$\int \frac{\cot^{-1}(x/a)}{x}\,dx = \frac{\pi}{2}\ln x - \int \frac{\tan^{-1}(x/a)}{x}\,dx$$ [See 17.24.16]

17.24.22 $$\int \frac{\cot^{-1}(x/a)}{x^2}\,dx = -\frac{\cot^{-1}(x/a)}{x} + \frac{1}{2a}\ln\left(\frac{x^2+a^2}{x^2}\right)$$

17.24.23 $$\int \sec^{-1}\frac{x}{a}\,dx = \begin{cases} x\sec^{-1}\dfrac{x}{a} - a\ln(x+\sqrt{x^2-a^2}) & 0<\sec^{-1}\dfrac{x}{a}<\dfrac{\pi}{2} \\ x\sec^{-1}\dfrac{x}{a} + a\ln(x+\sqrt{x^2-a^2}) & \dfrac{\pi}{2}<\sec^{-1}\dfrac{x}{a}<\pi \end{cases}$$

17.24.24 $$\int x\sec^{-1}\frac{x}{a}\,dx = \begin{cases} \dfrac{x^2}{2}\sec^{-1}\dfrac{x}{a} - \dfrac{a\sqrt{x^2-a^2}}{2} & 0<\sec^{-1}\dfrac{x}{a}<\dfrac{\pi}{2} \\ \dfrac{x^2}{2}\sec^{-1}\dfrac{x}{a} + \dfrac{a\sqrt{x^2-a^2}}{2} & \dfrac{\pi}{2}<\sec^{-1}\dfrac{x}{a}<\pi \end{cases}$$

17.24.25 $$\int x^2\sec^{-1}\frac{x}{a}\,dx = \begin{cases} \dfrac{x^3}{3}\sec^{-1}\dfrac{x}{a} - \dfrac{ax\sqrt{x^2-a^2}}{6} - \dfrac{a^3}{6}\ln(x+\sqrt{x^2-a^2}) & 0<\sec^{-1}\dfrac{x}{a}<\dfrac{\pi}{2} \\ \dfrac{x^3}{3}\sec^{-1}\dfrac{x}{a} + \dfrac{ax\sqrt{x^2-a^2}}{6} + \dfrac{a^3}{6}\ln(x+\sqrt{x^2-a^2}) & \dfrac{\pi}{2}<\sec^{-1}\dfrac{x}{a}<\pi \end{cases}$$

17.24.26 $$\int \frac{\sec^{-1}(x/a)}{x}\,dx = \frac{\pi}{2}\ln x + \frac{a}{x} + \frac{(a/x)^3}{2\cdot3\cdot3} + \frac{1\cdot3(a/x)^5}{2\cdot4\cdot5\cdot5} + \frac{1\cdot3\cdot5(a/x)^7}{2\cdot4\cdot6\cdot7\cdot7} + \cdots$$

17.24.27 $$\int \frac{\sec^{-1}(x/a)}{x^2}\,dx = \begin{cases} -\dfrac{\sec^{-1}(x/a)}{x} + \dfrac{\sqrt{x^2-a^2}}{ax} & 0<\sec^{-1}\dfrac{x}{a}<\dfrac{\pi}{2} \\ -\dfrac{\sec^{-1}(x/a)}{x} - \dfrac{\sqrt{x^2-a^2}}{ax} & \dfrac{\pi}{2}<\sec^{-1}\dfrac{x}{a}<\pi \end{cases}$$

17.24.28 $$\int \csc^{-1}\frac{x}{a}\,dx = \begin{cases} x\csc^{-1}\dfrac{x}{a} + a\ln(x+\sqrt{x^2-a^2}) & 0<\csc^{-1}\dfrac{x}{a}<\dfrac{\pi}{2} \\ x\csc^{-1}\dfrac{x}{a} - a\ln(x+\sqrt{x^2-a^2}) & -\dfrac{\pi}{2}<\csc^{-1}\dfrac{x}{a}<0 \end{cases}$$

17.24.29 $$\int x\csc^{-1}\frac{x}{a}\,dx = \begin{cases} \dfrac{x^2}{2}\csc^{-1}\dfrac{x}{a} + \dfrac{a\sqrt{x^2-a^2}}{2} & 0<\csc^{-1}\dfrac{x}{a}<\dfrac{\pi}{2} \\ \dfrac{x^2}{2}\csc^{-1}\dfrac{x}{a} - \dfrac{a\sqrt{x^2-a^2}}{2} & -\dfrac{\pi}{2}<\csc^{-1}\dfrac{x}{a}<0 \end{cases}$$

17.24.30 $$\int x^2\csc^{-1}\frac{x}{a}\,dx = \begin{cases} \dfrac{x^3}{3}\csc^{-1}\dfrac{x}{a} + \dfrac{ax\sqrt{x^2-a^2}}{6} + \dfrac{a^3}{6}\ln(x+\sqrt{x^2-a^2}) & 0<\csc^{-1}\dfrac{x}{a}<\dfrac{\pi}{2} \\ \dfrac{x^3}{3}\csc^{-1}\dfrac{x}{a} - \dfrac{ax\sqrt{x^2-a^2}}{6} - \dfrac{a^3}{6}\ln(x+\sqrt{x^2-a^2}) & -\dfrac{\pi}{2}<\csc^{-1}\dfrac{x}{a}<0 \end{cases}$$

17.24.31 $$\int \frac{\csc^{-1}(x/a)}{x}\,dx = -\left(\frac{a}{x} + \frac{(a/x)^3}{2\cdot3\cdot3} + \frac{1\cdot3(a/x)^5}{2\cdot4\cdot5\cdot5} + \frac{1\cdot3\cdot5(a/x)^7}{2\cdot4\cdot6\cdot7\cdot7} + \cdots\right)$$

17.24.32 $$\int \frac{\csc^{-1}(x/a)}{x^2}\,dx = \begin{cases} -\dfrac{\csc^{-1}(x/a)}{x} - \dfrac{\sqrt{x^2-a^2}}{ax} & 0 < \csc^{-1}\dfrac{x}{a} < \dfrac{\pi}{2} \\ -\dfrac{\csc^{-1}(x/a)}{x} + \dfrac{\sqrt{x^2-a^2}}{ax} & -\dfrac{\pi}{2} < \csc^{-1}\dfrac{x}{a} < 0 \end{cases}$$

17.24.33 $$\int x^m \sin^{-1}\frac{x}{a}\,dx = \frac{x^{m+1}}{m+1}\sin^{-1}\frac{x}{a} - \frac{1}{m+1}\int \frac{x^{m+1}}{\sqrt{a^2-x^2}}\,dx$$

17.24.34 $$\int x^m \cos^{-1}\frac{x}{a}\,dx = \frac{x^{m+1}}{m+1}\cos^{-1}\frac{x}{a} + \frac{1}{m+1}\int \frac{x^{m+1}}{\sqrt{a^2-x^2}}\,dx$$

17.24.35 $$\int x^m \tan^{-1}\frac{x}{a}\,dx = \frac{x^{m+1}}{m+1}\tan^{-1}\frac{x}{a} - \frac{a}{m+1}\int \frac{x^{m+1}}{x^2+a^2}\,dx$$

17.24.36 $$\int x^m \cot^{-1}\frac{x}{a}\,dx = \frac{x^{m+1}}{m+1}\cot^{-1}\frac{x}{a} + \frac{a}{m+1}\int \frac{x^{m+1}}{x^2+a^2}\,dx$$

17.24.37 $$\int x^m \sec^{-1}\frac{x}{a}\,dx = \begin{cases} \dfrac{x^{m+1}\sec^{-1}(x/a)}{m+1} - \dfrac{a}{m+1}\displaystyle\int \frac{x^m\,dx}{\sqrt{x^2-a^2}} & 0 < \sec^{-1}\dfrac{x}{a} < \dfrac{\pi}{2} \\ \dfrac{x^{m+1}\sec^{-1}(x/a)}{m+1} + \dfrac{a}{m+1}\displaystyle\int \frac{x^m\,dx}{\sqrt{x^2-a^2}} & \dfrac{\pi}{2} < \sec^{-1}\dfrac{x}{a} < \pi \end{cases}$$

17.24.38 $$\int x^m \csc^{-1}\frac{x}{a}\,dx = \begin{cases} \dfrac{x^{m+1}\csc^{-1}(x/a)}{m+1} + \dfrac{a}{m+1}\displaystyle\int \frac{x^m\,dx}{\sqrt{x^2-a^2}} & 0 < \csc^{-1}\dfrac{x}{a} < \dfrac{\pi}{2} \\ \dfrac{x^{m+1}\csc^{-1}(x/a)}{m+1} - \dfrac{a}{m+1}\displaystyle\int \frac{x^m\,dx}{\sqrt{x^2-a^2}} & -\dfrac{\pi}{2} < \csc^{-1}\dfrac{x}{a} < 0 \end{cases}$$

(25) INTEGRALS INVOLVING e^{ax}

17.25.1 $$\int e^{ax}\,dx = \frac{e^{ax}}{a}$$

17.25.2 $$\int xe^{ax}\,dx = \frac{e^{ax}}{a}\left(x - \frac{1}{a}\right)$$

17.25.3 $$\int x^2e^{ax}\,dx = \frac{e^{ax}}{a}\left(x^2 - \frac{2x}{a} + \frac{2}{a^2}\right)$$

17.25.4 $$\int x^n e^{ax}\,dx = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx$$

$$= \frac{e^{ax}}{a}\left(x^n - \frac{nx^{n-1}}{a} + \frac{n(n-1)x^{n-2}}{a^2} - \cdots \frac{(-1)^n n!}{a^n}\right) \quad \text{if } n = \text{positive integer}$$

17.25.5 $$\int \frac{e^{ax}}{x}\,dx = \ln x + \frac{ax}{1\cdot 1!} + \frac{(ax)^2}{2\cdot 2!} + \frac{(ax)^3}{3\cdot 3!} + \cdots$$

17.25.6 $$\int \frac{e^{ax}}{x^n}\,dx = \frac{-e^{ax}}{(n-1)x^{n-1}} + \frac{a}{n-1}\int \frac{e^{ax}}{x^{n-1}}\,dx$$

17.25.7 $$\int \frac{dx}{p+qe^{ax}} = \frac{x}{p} - \frac{1}{ap}\ln(p+qe^{ax})$$

17.25.8 $\displaystyle\int \frac{dx}{(p+qe^{ax})^2} = \frac{x}{p^2} + \frac{1}{ap(p+qe^{ax})} - \frac{1}{ap^2}\ln(p+qe^{ax})$

17.25.9 $\displaystyle\int \frac{dx}{pe^{ax}+qe^{-ax}} = \begin{cases} \dfrac{1}{a\sqrt{pq}}\tan^{-1}\left(\sqrt{\dfrac{p}{q}}\,e^{ax}\right) \\ \dfrac{1}{2a\sqrt{-pq}}\ln\left(\dfrac{e^{ax}-\sqrt{-q/p}}{e^{ax}+\sqrt{-q/p}}\right) \end{cases}$

17.25.10 $\displaystyle\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$

17.25.11 $\displaystyle\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$

17.25.12 $\displaystyle\int xe^{ax}\sin bx\,dx = \frac{xe^{ax}(a\sin bx - b\cos bx)}{a^2+b^2} - \frac{e^{ax}\{(a^2-b^2)\sin bx - 2ab\cos bx\}}{(a^2+b^2)^2}$

17.25.13 $\displaystyle\int xe^{ax}\cos bx\,dx = \frac{xe^{ax}(a\cos bx + b\sin bx)}{a^2+b^2} - \frac{e^{ax}\{(a^2-b^2)\cos bx - 2ab\sin bx\}}{(a^2+b^2)^2}$

17.25.14 $\displaystyle\int e^{ax}\ln x\,dx = \frac{e^{ax}\ln x}{a} - \frac{1}{a}\int \frac{e^{ax}}{x}\,dx$

17.25.15 $\displaystyle\int e^{ax}\sin^n bx\,dx = \frac{e^{ax}\sin^{n-1}bx}{a^2+n^2b^2}(a\sin bx - nb\cos bx) + \frac{n(n-1)b^2}{a^2+n^2b^2}\int e^{ax}\sin^{n-2}bx\,dx$

17.25.16 $\displaystyle\int e^{ax}\cos^n bx\,dx = \frac{e^{ax}\cos^{n-1}bx}{a^2+n^2b^2}(a\cos bx + nb\sin bx) + \frac{n(n-1)b^2}{a^2+n^2b^2}\int e^{ax}\cos^{n-2}bx\,dx$

(26) INTEGRALS INVOLVING $\ln x$

17.26.1 $\displaystyle\int \ln x\,dx = x\ln x - x$

17.26.2 $\displaystyle\int x\ln x\,dx = \frac{x^2}{2}(\ln x - \tfrac{1}{2})$

17.26.3 $\displaystyle\int x^m\ln x\,dx = \frac{x^{m+1}}{m+1}\left(\ln x - \frac{1}{m+1}\right)$ [If $m=-1$ see 17.26.4.]

17.26.4 $\displaystyle\int \frac{\ln x}{x}\,dx = \frac{1}{2}\ln^2 x$

17.26.5 $\displaystyle\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x} - \frac{1}{x}$

17.26.6 $\displaystyle\int \ln^2 x\,dx = x\ln^2 x - 2x\ln x + 2x$

17.26.7 $\displaystyle\int \frac{\ln^n x\,dx}{x} = \frac{\ln^{n+1}x}{n+1}$ [If $n=-1$ see 17.26.8.]

17.26.8 $\displaystyle\int \frac{dx}{x\ln x} = \ln(\ln x)$

17.26.9 $\displaystyle\int \frac{dx}{\ln x} = \ln(\ln x) + \ln x + \frac{\ln^2 x}{2\cdot 2!} + \frac{\ln^3 x}{3\cdot 3!} + \cdots$

17.26.10 $\displaystyle\int \frac{x^m\,dx}{\ln x} = \ln(\ln x) + (m+1)\ln x + \frac{(m+1)^2 \ln^2 x}{2\cdot 2!} + \frac{(m+1)^3 \ln^3 x}{3\cdot 3!} + \cdots$

17.26.11 $\displaystyle\int \ln^n x\,dx = x\ln^n x - n\int \ln^{n-1} x\,dx$

17.26.12 $\displaystyle\int x^m \ln^n x\,dx = \frac{x^{m+1}\ln^n x}{m+1} - \frac{n}{m+1}\int x^m \ln^{n-1} x\,dx$

If $m = -1$ see 17.26.7.

17.26.13 $\displaystyle\int \ln(x^2+a^2)\,dx = x\ln(x^2+a^2) - 2x + 2a\tan^{-1}\frac{x}{a}$

17.26.14 $\displaystyle\int \ln(x^2-a^2)\,dx = x\ln(x^2-a^2) - 2x + a\ln\left(\frac{x+a}{x-a}\right)$

17.26.15 $\displaystyle\int x^m \ln(x^2 \pm a^2)\,dx = \frac{x^{m+1}\ln(x^2 \pm a^2)}{m+1} - \frac{2}{m+1}\int \frac{x^{m+2}}{x^2 \pm a^2}\,dx$

(27) INTEGRALS INVOLVING [illegible]

17.27.1 $\displaystyle\int \sinh ax\,dx = \frac{\cosh ax}{a}$

17.27.2 $\displaystyle\int x\sinh ax\,dx = \frac{x\cosh ax}{a} - \frac{\sinh ax}{a^2}$

17.27.3 $\displaystyle\int x^2 \sinh ax\,dx = \left(\frac{x^2}{a} + \frac{2}{a^3}\right)\cosh ax - \frac{2x}{a^2}\sinh ax$

17.27.4 $\displaystyle\int \frac{\sinh ax}{x}\,dx = ax + \frac{(ax)^3}{3\cdot 3!} + \frac{(ax)^5}{5\cdot 5!} + \cdots$

17.27.5 $\displaystyle\int \frac{\sinh ax}{x^2}\,dx = -\frac{\sinh ax}{x} + a\int \frac{\cosh ax}{x}\,dx$ [See 17.28.4]

17.27.6 $\displaystyle\int \frac{dx}{\sinh ax} = \frac{1}{a}\ln\tanh\frac{ax}{2}$

17.27.7 $\displaystyle\int \frac{x\,dx}{\sinh ax} = \frac{1}{a^2}\left\{ax - \frac{(ax)^3}{18} + \frac{7(ax)^5}{1800} - \cdots + \frac{2(-1)^n(2^{2n}-1)B_n(ax)^{2n+1}}{(2n+1)!} + \cdots\right\}$

17.27.8 $\displaystyle\int \sinh^2 ax\,dx = \frac{\sinh ax\cosh ax}{2a} - \frac{x}{2}$

17.27.9 $\displaystyle\int x\sinh^2 ax\,dx = \frac{x\sinh 2ax}{4a} - \frac{\cosh 2ax}{8a^2} - \frac{x^2}{4}$

17.27.10 $\displaystyle\int \frac{dx}{\sinh^2 ax} = -\frac{\coth ax}{a}$

17.27.11 $\displaystyle\int \sinh ax \sinh px\,dx = \frac{\sinh(a+p)x}{2(a+p)} - \frac{\sinh(a-p)x}{2(a-p)}$

For $a = \pm p$ see 17.27.8.

17.27.12 $\displaystyle\int x^m \sinh ax\,dx = \frac{x^m \cosh ax}{a} - \frac{m}{a}\int x^{m-1}\cosh ax\,dx$ [See 17.28.12]

17.27.13 $\displaystyle\int \sinh^n ax\,dx = \frac{\sinh^{n-1} ax \cosh ax}{an} - \frac{n-1}{n}\int \sinh^{n-2} ax\,dx$

17.27.14 $\displaystyle\int \frac{\sinh ax}{x^n}dx = \frac{-\sinh ax}{(n-1)x^{n-1}} + \frac{a}{n-1}\int \frac{\cosh ax}{x^{n-1}}dx$ [See 17.28.14]

17.27.15 $\displaystyle\int \frac{dx}{\sinh^n ax} = \frac{-\cosh ax}{a(n-1)\sinh^{n-1} ax} - \frac{n-2}{n-1}\int \frac{dx}{\sinh^{n-2} ax}$

17.27.16 $\displaystyle\int \frac{x\,dx}{\sinh^n ax} = \frac{-x\cosh ax}{a(n-1)\sinh^{n-1} ax} - \frac{1}{a^2(n-1)(n-2)\sinh^{n-2} ax} - \frac{n-2}{n-1}\int \frac{x\,dx}{\sinh^{n-2} ax}$

(28) INTEGRALS INVOLVING $\cosh ax$

17.28.1 $\displaystyle\int \cosh ax\,dx = \frac{\sinh ax}{a}$

17.28.2 $\displaystyle\int x\cosh ax\,dx = \frac{x\sinh ax}{a} - \frac{\cosh ax}{a^2}$

17.28.3 $\displaystyle\int x^2\cosh ax\,dx = -\frac{2x\cosh ax}{a^2} + \left(\frac{x^2}{a} + \frac{2}{a^3}\right)\sinh ax$

17.28.4 $\displaystyle\int \frac{\cosh ax}{x}dx = \ln x + \frac{(ax)^2}{2\cdot 2!} + \frac{(ax)^4}{4\cdot 4!} + \frac{(ax)^6}{6\cdot 6!} + \cdots$

17.28.5 $\displaystyle\int \frac{\cosh ax}{x^2}dx = -\frac{\cosh ax}{x} + a\int \frac{\sinh ax}{x}dx$ [See 17.27.4]

17.28.6 $\displaystyle\int \frac{dx}{\cosh ax} = \frac{2}{a}\tan^{-1} e^{ax}$

17.28.7 $\displaystyle\int \frac{x\,dx}{\cosh ax} = \frac{1}{a^2}\left\{\frac{(ax)^2}{2} - \frac{(ax)^4}{8} + \frac{5(ax)^6}{144} + \cdots + \frac{(-1)^n E_n (ax)^{2n+2}}{(2n+2)(2n)!} + \cdots\right\}$

17.28.8 $\displaystyle\int \cosh^2 ax\,dx = \frac{x}{2} + \frac{\sinh ax \cosh ax}{2a}$

17.28.9 $\displaystyle\int x\cosh^2 ax\,dx = \frac{x^2}{4} + \frac{x\sinh 2ax}{4a} - \frac{\cosh 2ax}{8a^2}$

17.28.10 $\displaystyle\int \frac{dx}{\cosh^2 ax} = \frac{\tanh ax}{a}$

17.28.11 $\displaystyle\int \cosh ax \cosh px\, dx = \frac{\sinh(a-p)x}{2(a-p)} + \frac{\sinh(a+p)x}{2(a+p)}$

17.28.12 $\displaystyle\int x^m \cosh ax\, dx = \frac{x^m \sinh ax}{a} - \frac{m}{a}\int x^{m-1}\sinh ax\, dx$ [See 17.27.12]

17.28.13 $\displaystyle\int \cosh^n ax\, dx = \frac{\cosh^{n-1} ax \sinh ax}{an} + \frac{n-1}{n}\int \cosh^{n-2} ax\, dx$

17.28.14 $\displaystyle\int \frac{\cosh ax}{x^n}\, dx = \frac{-\cosh ax}{(n-1)x^{n-1}} + \frac{a}{n-1}\int \frac{\sinh ax}{x^{n-1}}\, dx$ [See 17.27.14]

17.28.15 $\displaystyle\int \frac{dx}{\cosh^n ax} = \frac{\sinh ax}{a(n-1)\cosh^{n-1} ax} + \frac{n-2}{n-1}\int \frac{dx}{\cosh^{n-2} ax}$

17.28.16 $\displaystyle\int \frac{x\, dx}{\cosh^n ax} = \frac{x \sinh ax}{a(n-1)\cosh^{n-1} ax} + \frac{1}{(n-1)(n-2)a^2\cosh^{n-2} ax} + \frac{n-2}{n-1}\int \frac{x\, dx}{\cosh^{n-2} ax}$

(29) INTEGRALS INVOLVING $\sinh ax$ AND $\cosh ax$

17.29.1 $\displaystyle\int \sinh ax \cosh ax\, dx = \frac{\sinh^2 ax}{2a}$

17.29.2 $\displaystyle\int \sinh px \cosh qx\, dx = \frac{\cosh(p+q)x}{2(p+q)} + \frac{\cosh(p-q)x}{2(p-q)}$

17.29.3 $\displaystyle\int \sinh^2 ax \cosh^2 ax\, dx = \frac{\sinh 4ax}{32a} - \frac{x}{8}$

17.29.4 $\displaystyle\int \frac{dx}{\sinh ax \cosh ax} = \frac{1}{a}\ln \tanh ax$

17.29.5 $\displaystyle\int \frac{dx}{\sinh^2 ax \cosh^2 ax} = -\frac{2\coth 2ax}{a}$

17.29.6 $\displaystyle\int \frac{\sinh^2 ax}{\cosh ax}\, dx = \frac{\sinh ax}{a} - \frac{1}{a}\tan^{-1}\sinh ax$

17.29.7 $\displaystyle\int \frac{\cosh^2 ax}{\sinh ax}\, dx = \frac{\cosh ax}{a} + \frac{1}{a}\ln\tanh\frac{ax}{2}$

(30) INTEGRALS INVOLVING tanh ax

17.30.1 $$\int \tanh ax\,dx = \frac{1}{a}\ln\cosh ax$$

17.30.2 $$\int \tanh^2 ax\,dx = x - \frac{\tanh ax}{a}$$

17.30.3 $$\int \tanh^3 ax\,dx = \frac{1}{a}\ln\cosh ax - \frac{\tanh^2 ax}{2a}$$

17.30.4 $$\int x\tanh ax\,dx = \frac{1}{a^2}\left\{\frac{(ax)^3}{3} - \frac{(ax)^5}{15} + \frac{2(ax)^7}{105} - \cdots \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_n(ax)^{2n+1}}{(2n+1)!} + \cdots\right\}$$

17.30.5 $$\int x\tanh^2 ax\,dx = \frac{x^2}{2} - \frac{x\tanh ax}{a} + \frac{1}{a^2}\ln\cosh ax$$

17.30.6 $$\int \frac{\tanh ax}{x}\,dx = ax - \frac{(ax)^3}{9} + \frac{2(ax)^5}{75} - \cdots \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_n(ax)^{2n-1}}{(2n-1)(2n)!} + \cdots$$

17.30.7 $$\int \frac{dx}{p + q\tanh ax} = \frac{px}{p^2 - q^2} - \frac{q}{a(p^2 - q^2)}\ln(q\sinh ax + p\cosh ax)$$

17.30.8 $$\int \tanh^n ax\,dx = \frac{-\tanh^{n-1} ax}{a(n-1)} + \int \tanh^{n-2} ax\,dx$$

(31) INTEGRALS INVOLVING coth ax

17.31.1 $$\int \coth ax\,dx = \frac{1}{a}\ln\sinh ax$$

17.31.2 $$\int \coth^2 ax\,dx = x - \frac{\coth ax}{a}$$

17.31.3 $$\int \coth^3 ax\,dx = \frac{1}{a}\ln\sinh ax - \frac{\coth^2 ax}{2a}$$

17.31.4 $$\int x\coth ax\,dx = \frac{1}{a^2}\left\{ax + \frac{(ax)^3}{9} - \frac{(ax)^5}{225} + \cdots \frac{(-1)^{n-1}2^{2n}B_n(ax)^{2n+1}}{(2n+1)!} + \cdots\right\}$$

17.31.5 $$\int x\coth^2 ax\,dx = \frac{x^2}{2} - \frac{x\coth ax}{a} + \frac{1}{a^2}\ln\sinh ax$$

17.31.6 $$\int \frac{\coth ax}{x}\,dx = -\frac{1}{ax} + \frac{ax}{3} - \frac{(ax)^3}{135} + \cdots \frac{(-1)^n 2^{2n}B_n(ax)^{2n-1}}{(2n-1)(2n)!} + \cdots$$

17.31.7 $$\int \frac{dx}{p + q\coth ax} = \frac{px}{p^2 - q^2} - \frac{q}{a(p^2 - q^2)}\ln(p\sinh ax + q\cosh ax)$$

17.31.8 $$\int \coth^n ax\,dx = -\frac{\coth^{n-1} ax}{a(n-1)} + \int \coth^{n-2} ax\,dx$$

(32) INTEGRALS INVOLVING sech *ax*

17.32.1 $\displaystyle\int \operatorname{sech} ax\, dx = \frac{2}{a}\tan^{-1} e^{ax}$

17.32.2 $\displaystyle\int \operatorname{sech}^2 ax\, dx = \frac{\tanh ax}{a}$

17.32.3 $\displaystyle\int \operatorname{sech}^3 ax\, dx = \frac{\operatorname{sech} ax \tanh ax}{2a} + \frac{1}{2a}\tan^{-1}\sinh ax$

17.32.4 $\displaystyle\int x \operatorname{sech} ax\, dx = \frac{1}{a^2}\left\{\frac{(ax)^2}{2} - \frac{(ax)^4}{8} + \frac{5(ax)^6}{144} + \cdots \frac{(-1)^n E_n (ax)^{2n+2}}{(2n+2)(2n)!} + \cdots\right\}$

17.32.5 $\displaystyle\int x \operatorname{sech}^2 ax\, dx = \frac{x \tanh ax}{a} - \frac{1}{a^2}\ln\cosh ax$

17.32.6 $\displaystyle\int \frac{\operatorname{sech} ax}{x}\, dx = \ln x - \frac{(ax)^2}{4} + \frac{5(ax)^4}{96} - \frac{61(ax)^6}{4320} + \cdots \frac{(-1)^n E_n (ax)^{2n}}{2n(2n)!} + \cdots$

17.32.7 $\displaystyle\int \operatorname{sech}^n ax\, dx = \frac{\operatorname{sech}^{n-2} ax \tanh ax}{a(n-1)} + \frac{n-2}{n-1}\int \operatorname{sech}^{n-2} ax\, dx$

(33) INTEGRALS INVOLVING csch *ax*

17.33.1 $\displaystyle\int \operatorname{csch} ax\, dx = \frac{1}{a}\ln\tanh\frac{ax}{2}$

17.33.2 $\displaystyle\int \operatorname{csch}^2 ax\, dx = -\frac{\coth ax}{a}$

17.33.3 $\displaystyle\int \operatorname{csch}^3 ax\, dx = -\frac{\operatorname{csch} ax \coth ax}{2a} - \frac{1}{2a}\ln\tanh\frac{ax}{2}$

17.33.4 $\displaystyle\int x \operatorname{csch} ax\, dx = \frac{1}{a^2}\left\{ax - \frac{(ax)^3}{18} + \frac{7(ax)^5}{1800} + \cdots + \frac{2(-1)^n(2^{2n-1}-1)B_n(ax)^{2n+1}}{(2n+1)!} + \cdots\right\}$

17.33.5 $\displaystyle\int x \operatorname{csch}^2 ax\, dx = -\frac{x\coth ax}{a} + \frac{1}{a^2}\ln\sinh ax$

17.33.6 $\displaystyle\int \frac{\operatorname{csch} ax}{x}\, dx = -\frac{1}{ax} - \frac{ax}{6} + \frac{7(ax)^3}{1080} + \cdots \frac{(-1)^n 2(2^{2n-1}-1)B_n(ax)^{2n-1}}{(2n-1)(2n)!} + \cdots$

17.33.7 $\displaystyle\int \operatorname{csch}^n ax\, dx = \frac{-\operatorname{csch}^{n-2} ax \coth ax}{a(n-1)} - \frac{n-2}{n-1}\int \operatorname{csch}^{n-2} ax\, dx$

(34) INTEGRALS INVOLVING INVERSE HYPERBOLIC FUNCTIONS

17.34.1 $$\int \sinh^{-1}\frac{x}{a}\,dx = x\sinh^{-1}\frac{x}{a} - \sqrt{x^2+a^2}$$

17.34.2 $$\int x\sinh^{-1}\frac{x}{a}\,dx = \left(\frac{x^2}{2}+\frac{a^2}{4}\right)\sinh^{-1}\frac{x}{a} - \frac{x\sqrt{x^2+a^2}}{4}$$

17.34.3 $$\int \frac{\sinh^{-1}(x/a)}{x}\,dx = \begin{cases} \dfrac{x}{a} - \dfrac{(x/a)^3}{2\cdot3\cdot3} + \dfrac{1\cdot3(x/a)^5}{2\cdot4\cdot5\cdot5} - \dfrac{1\cdot3\cdot5(x/a)^7}{2\cdot4\cdot6\cdot7\cdot7} + \cdots & |x|<a \\ \dfrac{\ln^2(2x/a)}{2} - \dfrac{(a/x)^2}{2\cdot2\cdot2} + \dfrac{1\cdot3(a/x)^4}{2\cdot4\cdot4\cdot4} - \dfrac{1\cdot3\cdot5(a/x)^6}{2\cdot4\cdot6\cdot6\cdot6} + \cdots & x>a \\ -\dfrac{\ln^2(-2x/a)}{2} + \dfrac{(a/x)^2}{2\cdot2\cdot2} - \dfrac{1\cdot3(a/x)^4}{2\cdot4\cdot4\cdot4} + \dfrac{1\cdot3\cdot5(a/x)^6}{2\cdot4\cdot6\cdot6\cdot6} - \cdots & x<-a \end{cases}$$

17.34.4 $$\int \cosh^{-1}\frac{x}{a}\,dx = \begin{cases} x\cosh^{-1}(x/a) - \sqrt{x^2-a^2}, & \cosh^{-1}(x/a)>0 \\ x\cosh^{-1}(x/a) + \sqrt{x^2-a^2}, & \cosh^{-1}(x/a)<0 \end{cases}$$

17.34.5 $$\int x\cosh^{-1}\frac{x}{a}\,dx = \begin{cases} \frac{1}{4}(2x^2-a^2)\cosh^{-1}(x/a) - \frac{1}{4}x\sqrt{x^2-a^2}, & \cosh^{-1}(x/a)>0 \\ \frac{1}{4}(2x^2-a^2)\cosh^{-1}(x/a) + \frac{1}{4}x\sqrt{x^2-a^2}, & \cosh^{-1}(x/a)<0 \end{cases}$$

17.34.6 $$\int \frac{\cosh^{-1}(x/a)}{x}\,dx = \pm\left[\frac{1}{2}\ln^2(2x/a) + \frac{(a/x)^2}{2\cdot2\cdot2} + \frac{1\cdot3(a/x)^4}{2\cdot4\cdot4\cdot4} + \frac{1\cdot3\cdot5(a/x)^6}{2\cdot4\cdot6\cdot6\cdot6} + \cdots\right]$$
$+$ if $\cosh^{-1}(x/a)>0$, $-$ if $\cosh^{-1}(x/a)<0$

17.34.7 $$\int \tanh^{-1}\frac{x}{a}\,dx = x\tanh^{-1}\frac{x}{a} + \frac{a}{2}\ln(a^2-x^2)$$

17.34.8 $$\int x\tanh^{-1}\frac{x}{a}\,dx = \frac{ax}{2} + \tfrac{1}{2}(x^2-a^2)\tanh^{-1}\frac{x}{a}$$

17.34.9 $$\int \frac{\tanh^{-1}(x/a)}{x}\,dx = \frac{x}{a} + \frac{(x/a)^3}{3^2} + \frac{(x/a)^5}{5^2} + \cdots$$

17.34.10 $$\int \coth^{-1}\frac{x}{a}\,dx = x\coth^{-1}x + \frac{a}{2}\ln(x^2-a^2)$$

17.34.11 $$\int x\coth^{-1}\frac{x}{a}\,dx = \frac{ax}{2} + \tfrac{1}{2}(x^2-a^2)\coth^{-1}\frac{x}{a}$$

17.34.12 $$\int \frac{\coth^{-1}(x/a)}{x}\,dx = -\left(\frac{a}{x} + \frac{(a/x)^3}{3^2} + \frac{(a/x)^5}{5^2} + \cdots\right)$$

17.34.13 $$\int \operatorname{sech}^{-1}\frac{x}{a}\,dx = \begin{cases} x\operatorname{sech}^{-1}(x/a) + a\sin^{-1}(x/a), & \operatorname{sech}^{-1}(x/a)>0 \\ x\operatorname{sech}^{-1}(x/a) - a\sin^{-1}(x/a), & \operatorname{sech}^{-1}(x/a)<0 \end{cases}$$

17.34.14 $$\int \operatorname{csch}^{-1}\frac{x}{a}\,dx = x\operatorname{csch}^{-1}\frac{x}{a} \pm a\sinh^{-1}\frac{x}{a} \qquad [+ \text{ if } x>0,\ - \text{ if } x<0]$$

17.34.15 $\int x^m \sinh^{-1}\frac{x}{a}\,dx = \frac{x^{m+1}}{m+1}\sinh^{-1}\frac{x}{a} - \frac{1}{m+1}\int \frac{x^{m+1}}{\sqrt{x^2+a^2}}\,dx$

17.34.16 $\int x^m \cosh^{-1}\frac{x}{a}\,dx = \begin{cases} \frac{x^{m+1}}{m+1}\cosh^{-1}\frac{x}{a} - \frac{1}{m+1}\int \frac{x^{m+1}}{\sqrt{x^2-a^2}}\,dx & \cosh^{-1}(x/a) > 0 \\ \frac{x^{m+1}}{m+1}\cosh^{-1}\frac{x}{a} + \frac{1}{m+1}\int \frac{x^{m+1}}{\sqrt{x^2-a^2}}\,dx & \cosh^{-1}(x/a) < 0 \end{cases}$

17.34.17 $\int x^m \tanh^{-1}\frac{x}{a}\,dx = \frac{x^{m+1}}{m+1}\tanh^{-1}\frac{x}{a} - \frac{a}{m+1}\int \frac{x^{m+1}}{a^2-x^2}\,dx$

17.34.18 $\int x^m \coth^{-1}\frac{x}{a}\,dx = \frac{x^{m+1}}{m+1}\coth^{-1}\frac{x}{a} - \frac{1}{m+1}\int \frac{x^{m+1}}{a^2-x^2}\,dx$

17.34.19 $\int x^m \operatorname{sech}^{-1}\frac{x}{a}\,dx = \begin{cases} \frac{x^{m+1}}{m+1}\operatorname{sech}^{-1}\frac{x}{a} + \frac{a}{m+1}\int \frac{x^m\,dx}{\sqrt{a^2-x^2}} & \operatorname{sech}^{-1}(x/a) > 0 \\ \frac{x^{m+1}}{m+1}\operatorname{sech}^{-1}\frac{x}{a} - \frac{1}{m+1}\int \frac{x^m\,dx}{\sqrt{a^2-x^2}} & \operatorname{sech}^{-1}(x/a) < 0 \end{cases}$

17.34.20 $\int x^m \operatorname{csch}^{-1}\frac{x}{a}\,dx = \frac{x^{m+1}}{m+1}\operatorname{csch}^{-1}\frac{x}{a} \pm \frac{a}{m+1}\int \frac{x^m\,dx}{\sqrt{x^2+a^2}}$ [+ if $x>0$, − if $x<0$]

18 DEFINITE INTEGRALS

DEFINITION OF A DEFINITE INTEGRAL

Let $f(x)$ be defined in an interval $a \leqq x \leqq b$. Divide the interval into n equal parts of length $\Delta x = (b-a)/n$. Then the definite integral of $f(x)$ between $x = a$ and $x = b$ is defined as

18.1 $$\int_a^b f(x)\,dx = \lim_{n\to\infty} \{f(a)\,\Delta x + f(a+\Delta x)\,\Delta x + f(a+2\Delta x)\,\Delta x + \cdots + f(a+(n-1)\,\Delta x)\,\Delta x\}$$

The limit will certainly exist if $f(x)$ is piecewise continuous.

If $f(x) = \frac{d}{dx}g(x)$, then by the fundamental theorem of the integral calculus the above definite integral can be evaluated by using the result

18.2 $$\int_a^b f(x)\,dx = \int_a^b \frac{d}{dx}g(x)\,dx = g(x)\Big|_a^b = g(b) - g(a)$$

If the interval is infinite or if $f(x)$ has a singularity at some point in the interval, the definite integral is called an *improper integral* and can be defined by using appropriate limiting procedures. For example,

18.3 $$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

18.4 $$\int_{-\infty}^\infty f(x)\,dx = \lim_{\substack{a\to-\infty \\ b\to\infty}} \int_a^b f(x)\,dx$$

18.5 $$\int_a^b f(x)\,dx = \lim_{\epsilon\to 0} \int_a^{b-\epsilon} f(x)\,dx \quad \text{if } b \text{ is a singular point.}$$

18.6 $$\int_a^b f(x)\,dx = \lim_{\epsilon\to 0} \int_{a+\epsilon}^{b} f(x)\,dx \quad \text{if } a \text{ is a singular point.}$$

GENERAL FORMULAS INVOLVING DEFINITE INTEGRALS

18.7 $$\int_a^b \{f(x) \pm g(x) \pm h(x) \pm \cdots\}\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx \pm \int_a^b h(x)\,dx \pm \cdots$$

18.8 $$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx \quad \text{where } c \text{ is any constant.}$$

18.9 $$\int_a^a f(x)\,dx = 0$$

18.10 $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

18.11 $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

18.12 $$\int_a^b f(x)\,dx = (b-a)f(c) \quad \text{where } c \text{ is between } a \text{ and } b.$$

This is called the *mean value theorem* for definite integrals and is valid if $f(x)$ is continuous in $a \leqq x \leqq b$.

18.13 $\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx$ where c is between a and b

This is a generalization of 18.12 and is valid if $f(x)$ and $g(x)$ are continuous in $a \leqq x \leqq b$ and $g(x) \geqq 0$.

LEIBNITZ'S RULE FOR DIFFERENTIATION OF INTEGRALS

18.14 $$\frac{d}{d\alpha}\int_{\phi_1(\alpha)}^{\phi_2(\alpha)} F(x,\alpha)\,dx = \int_{\phi_1(\alpha)}^{\phi_2(\alpha)} \frac{\partial F}{\partial \alpha}\,dx + F(\phi_2,\alpha)\frac{d\phi_2}{d\alpha} - F(\phi_1,\alpha)\frac{d\phi_1}{d\alpha}$$

APPROXIMATE FORMULAS FOR DEFINITE INTEGRALS

In the following the interval from $x = a$ to $x = b$ is subdivided into n equal parts by the points $a = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = b$ and we let $y_0 = f(x_0), y_1 = f(x_1), y_2 = f(x_2), \ldots, y_n = f(x_n), h = (b-a)/n$.

Rectangular formula:

18.15 $$\int_a^b f(x)\,dx \approx h(y_0 + y_1 + y_2 + \cdots + y_{n-1})$$

Trapezoidal formula:

18.16 $$\int_a^b f(x)\,dx \approx \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n)$$

Simpson's formula (or parabolic formula) for n even:

18.17 $$\int_a^b f(x)\,dx \approx \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n)$$

DEFINITE INTEGRALS INVOLVING RATIONAL OR IRRATIONAL EXPRESSIONS

18.18 $$\int_0^\infty \frac{dx}{x^2+a^2} = \frac{\pi}{2a}$$

18.19 $$\int_0^\infty \frac{x^{p-1}\,dx}{1+x} = \frac{\pi}{\sin p\pi}, \quad 0<p<1$$

18.20 $$\int_0^\infty \frac{x^m\,dx}{x^n+a^n} = \frac{\pi a^{m+1-n}}{n\sin[(m+1)\pi/n]}, \quad 0<m+1<n$$

18.21 $$\int_0^\infty \frac{x^m\,dx}{1+2x\cos\beta+x^2} = \frac{\pi}{\sin m\pi}\frac{\sin m\beta}{\sin\beta}$$

18.22 $$\int_0^a \frac{dx}{\sqrt{a^2-x^2}} = \frac{\pi}{2}$$

18.23 $$\int_0^a \sqrt{a^2-x^2}\,dx = \frac{\pi a^2}{4}$$

18.24 $$\int_0^a x^m(a^n-x^n)^p\,dx = \frac{a^{m+1+np}\,\Gamma[(m+1)/n]\,\Gamma(p+1)}{n\Gamma[(m+1)/n+p+1]}$$

18.25 $$\int_0^\infty \frac{x^m\,dx}{(x^n+a^n)^r} = \frac{(-1)^{r-1}\pi a^{m+1-nr}\Gamma[(m+1)/n]}{n\sin[(m+1)\pi/n](r-1)!\,\Gamma[(m+1)/n-r+1]}, \quad 0<m+1<nr$$

DEFINITE INTEGRALS INVOLVING TRIGONOMETRIC FUNCTIONS

All letters are considered positive unless otherwise indicated.

18.26 $$\int_0^\pi \sin mx \sin nx\,dx = \begin{cases} 0 & m,n \text{ integers and } m \neq n \\ \pi/2 & m,n \text{ integers and } m = n \end{cases}$$

18.27 $$\int_0^\pi \cos mx \cos nx\,dx = \begin{cases} 0 & m,n \text{ integers and } m \neq n \\ \pi/2 & m,n \text{ integers and } m = n \end{cases}$$

18.28 $$\int_0^\pi \sin mx \cos nx\,dx = \begin{cases} 0 & m,n \text{ integers and } m+n \text{ even} \\ 2m/(m^2-n^2) & m,n \text{ integers and } m+n \text{ odd} \end{cases}$$

18.29 $$\int_0^{\pi/2} \sin^2 x\,dx = \int_0^{\pi/2} \cos^2 x\,dx = \frac{\pi}{4}$$

18.30 $$\int_0^{\pi/2} \sin^{2m} x\,dx = \int_0^{\pi/2} \cos^{2m} x\,dx = \frac{1\cdot 3\cdot 5\cdots 2m-1}{2\cdot 4\cdot 6\cdots 2m}\frac{\pi}{2}, \quad m=1,2,\ldots$$

18.31 $$\int_0^{\pi/2} \sin^{2m+1} x\,dx = \int_0^{\pi/2} \cos^{2m+1} x\,dx = \frac{2\cdot 4\cdot 6\cdots 2m}{1\cdot 3\cdot 5\cdots 2m+1}, \quad m=1,2,\ldots$$

18.32 $$\int_0^{\pi/2} \sin^{2p-1} x \cos^{2q-1} x\,dx = \frac{\Gamma(p)\Gamma(q)}{2\Gamma(p+q)}$$

18.33 $$\int_0^\infty \frac{\sin px}{x}\,dx = \begin{cases} \pi/2 & p>0 \\ 0 & p=0 \\ -\pi/2 & p<0 \end{cases}$$

18.34 $$\int_0^\infty \frac{\sin px \cos qx}{x}\,dx = \begin{cases} 0 & p>q>0 \\ \pi/2 & 0<p<q \\ \pi/4 & p=q>0 \end{cases}$$

18.35 $$\int_0^\infty \frac{\sin px \sin qx}{x^2}\,dx = \begin{cases} \pi p/2 & 0<p\leqq q \\ \pi q/2 & p\geqq q>0 \end{cases}$$

18.36 $$\int_0^\infty \frac{\sin^2 px}{x^2}\,dx = \frac{\pi p}{2}$$

18.37 $$\int_0^\infty \frac{1-\cos px}{x^2}\,dx = \frac{\pi p}{2}$$

18.38 $$\int_0^\infty \frac{\cos px - \cos qx}{x}\,dx = \ln\frac{q}{p}$$

18.39 $$\int_0^\infty \frac{\cos px - \cos qx}{x^2}\,dx = \frac{\pi(q-p)}{2}$$

18.40 $$\int_0^\infty \frac{\cos mx}{x^2+a^2}\,dx = \frac{\pi}{2a}e^{-ma}$$

18.41 $$\int_0^\infty \frac{x\sin mx}{x^2+a^2}\,dx = \frac{\pi}{2}e^{-ma}$$

18.42 $$\int_0^\infty \frac{\sin mx}{x(x^2+a^2)}\,dx = \frac{\pi}{2a^2}(1-e^{-ma})$$

18.43 $$\int_0^{2\pi} \frac{dx}{a+b\sin x} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

18.44 $$\int_0^{2\pi} \frac{dx}{a+b\cos x} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

18.45 $$\int_0^{\pi/2} \frac{dx}{a+b\cos x} = \frac{\cos^{-1}(b/a)}{\sqrt{a^2-b^2}}$$

18.46 $$\int_0^{2\pi} \frac{dx}{(a+b\sin x)^2} = \int_0^{2\pi} \frac{dx}{(a+b\cos x)^2} = \frac{2\pi a}{(a^2-b^2)^{3/2}}$$

18.47 $$\int_0^{2\pi} \frac{dx}{1-2a\cos x + a^2} = \frac{2\pi}{1-a^2}, \qquad 0<a<1$$

18.48 $$\int_0^{\pi} \frac{x\sin x\,dx}{1-2a\cos x + a^2} = \begin{cases} (\pi/a)\ln(1+a), & |a|<1 \\ \pi\ln(1+1/a), & |a|>1 \end{cases}$$

18.49 $$\int_0^{\pi} \frac{\cos mx\,dx}{1-2a\cos x + a^2} = \frac{\pi a^m}{1-a^2}, \qquad a^2<1, \qquad m=0,1,2,\ldots$$

18.50 $$\int_0^{\infty} \sin ax^2\,dx = \int_0^{\infty} \cos ax^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}$$

18.51 $$\int_0^{\infty} \sin ax^n\,dx = \frac{1}{na^{1/n}}\Gamma(1/n)\sin\frac{\pi}{2n}, \qquad n>1$$

18.52 $$\int_0^{\infty} \cos ax^n\,dx = \frac{1}{na^{1/n}}\Gamma(1/n)\cos\frac{\pi}{2n}, \qquad n>1$$

18.53 $$\int_0^{\infty} \frac{\sin x}{\sqrt{x}}\,dx = \int_0^{\infty} \frac{\cos x}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{2}}$$

18.54 $$\int_0^{\infty} \frac{\sin x}{x^p}\,dx = \frac{\pi}{2\Gamma(p)\sin(p\pi/2)}, \qquad 0<p<1$$

18.55 $$\int_0^{\infty} \frac{\cos x}{x^p}\,dx = \frac{\pi}{2\Gamma(p)\cos(p\pi/2)}, \qquad 0<p<1$$

18.56 $$\int_0^{\infty} \sin ax^2 \cos 2bx\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}\left(\cos\frac{b^2}{a} - \sin\frac{b^2}{a}\right)$$

18.57 $$\int_0^{\infty} \cos ax^2 \cos 2bx\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}\left(\cos\frac{b^2}{a} + \sin\frac{b^2}{a}\right)$$

18.58 $$\int_0^{\infty} \frac{\sin^3 x}{x^3}\,dx = \frac{3\pi}{8}$$

18.59 $$\int_0^{\infty} \frac{\sin^4 x}{x^4}\,dx = \frac{\pi}{3}$$

18.60 $$\int_0^{\infty} \frac{\tan x}{x}\,dx = \frac{\pi}{2}$$

18.61 $$\int_0^{\pi/2} \frac{dx}{1+\tan^m x} = \frac{\pi}{4}$$

18.62 $$\int_0^{\pi/2} \frac{x}{\sin x}\,dx = 2\left\{\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots\right\}$$

18.63 $$\int_0^{1} \frac{\tan^{-1} x}{x}\,dx = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots$$

18.64 $$\int_0^{1} \frac{\sin^{-1} x}{x}\,dx = \frac{\pi}{2}\ln 2$$

18.65 $\displaystyle\int_0^1 \frac{1-\cos x}{x}\,dx - \int_1^\infty \frac{\cos x}{x}\,dx = \gamma$

18.66 $\displaystyle\int_0^\infty \left(\frac{1}{1+x^2} - \cos x\right)\frac{dx}{x} = \gamma$

18.67 $\displaystyle\int_0^\infty \frac{\tan^{-1} px - \tan^{-1} qx}{x}\,dx = \frac{\pi}{2}\ln\frac{p}{q}$

DEFINITE INTEGRALS INVOLVING EXPONENTIAL FUNCTIONS

Some integrals contain Euler's constant $\gamma = 0.5772156\ldots$ [see 1.20, page 3].

18.68 $\displaystyle\int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}$

18.69 $\displaystyle\int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}$

18.70 $\displaystyle\int_0^\infty \frac{e^{-ax}\sin bx}{x}\,dx = \tan^{-1}\frac{b}{a}$

18.71 $\displaystyle\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\,dx = \ln\frac{b}{a}$

18.72 $\displaystyle\int_0^\infty e^{-ax^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$

18.73 $\displaystyle\int_0^\infty e^{-ax^2}\cos bx\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\,e^{-b^2/4a}$

18.74 $\displaystyle\int_0^\infty e^{-(ax^2+bx+c)}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\,e^{(b^2-4ac)/4a}\operatorname{erfc}\frac{b}{2\sqrt{a}}$

where $\displaystyle\operatorname{erfc}(p) = \frac{2}{\sqrt{\pi}}\int_p^\infty e^{-x^2}\,dx$

18.75 $\displaystyle\int_{-\infty}^\infty e^{-(ax^2+bx+c)}\,dx = \sqrt{\frac{\pi}{a}}\,e^{(b^2-4ac)/4a}$

18.76 $\displaystyle\int_0^\infty x^n e^{-ax}\,dx = \frac{\Gamma(n+1)}{a^{n+1}}$

18.77 $\displaystyle\int_0^\infty x^m e^{-ax^2}\,dx = \frac{\Gamma[(m+1)/2]}{2a^{(m+1)/2}}$

18.78 $\displaystyle\int_0^\infty e^{-(ax^2+b/x^2)}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\,e^{-2\sqrt{ab}}$

18.79 $\displaystyle\int_0^\infty \frac{x\,dx}{e^x-1} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots = \frac{\pi^2}{6}$

18.80 $\displaystyle\int_0^\infty \frac{x^{n-1}}{e^x-1}\,dx = \Gamma(n)\left(\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\cdots\right)$

For even n this can be summed in terms of Bernoulli numbers [see pages 139 to 140].

18.81 $\displaystyle\int_0^\infty \frac{x\,dx}{e^x+1} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$

18.82 $\displaystyle\int_0^\infty \frac{x^{n-1}}{e^x+1}\,dx = \Gamma(n)\left(\frac{1}{1^n} - \frac{1}{2^n} + \frac{1}{3^n} - \cdots\right)$

For some positive integer values of n the series can be summed [see 23.10].

18.83 $\displaystyle\int_0^\infty \frac{\sin mx}{e^{2\pi x}-1}\,dx = \frac{1}{4}\coth\frac{m}{2} - \frac{1}{2m}$

18.84 $\displaystyle\int_0^\infty \left(\frac{1}{1+x} - e^{-x}\right)\frac{dx}{x} = \gamma$

18.85 $\displaystyle\int_0^\infty \frac{e^{-x^2} - e^{-x}}{x}\,dx = \tfrac{1}{2}\gamma$

18.86 $\displaystyle\int_0^\infty \left(\frac{1}{e^x-1} - \frac{e^{-x}}{x}\right)dx = \gamma$

18.87 $\displaystyle\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x\sec px}\,dx = \frac{1}{2}\ln\left(\frac{b^2+p^2}{a^2+p^2}\right)$

18.88 $\displaystyle\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x\csc px}\,dx = \tan^{-1}\frac{b}{p} - \tan^{-1}\frac{a}{p}$

18.89 $\displaystyle\int_0^\infty \frac{e^{-ax}(1-\cos x)}{x^2}\,dx = \cot^{-1}a - \frac{a}{2}\ln(a^2+1)$

DEFINITE INTEGRALS INVOLVING LOGARITHMIC FUNCTIONS

18.90 $\displaystyle\int_0^1 x^m(\ln x)^n\,dx = \frac{(-1)^n n!}{(m+1)^{n+1}} \qquad m > -1,\quad n = 0, 1, 2, \ldots$

If $n \neq 0, 1, 2, \ldots$ replace $n!$ by $\Gamma(n+1)$.

18.91 $\displaystyle\int_0^1 \frac{\ln x}{1+x}\,dx = -\frac{\pi^2}{12}$

18.92 $\displaystyle\int_0^1 \frac{\ln x}{1-x}\,dx = -\frac{\pi^2}{6}$

18.93 $\displaystyle\int_0^1 \frac{\ln(1+x)}{x}\,dx = \frac{\pi^2}{12}$

18.94 $\displaystyle\int_0^1 \frac{\ln(1-x)}{x}\,dx = -\frac{\pi^2}{6}$

18.95 $\displaystyle\int_0^1 \ln x\ln(1+x)\,dx = 2 - 2\ln 2 - \frac{\pi^2}{12}$

18.96 $\displaystyle\int_0^1 \ln x \ln(1-x)\,dx = 2 - \frac{\pi^2}{6}$

18.97 $\displaystyle\int_0^\infty \frac{x^{p-1}\ln x}{1+x}\,dx = -\pi^2 \csc p\pi \cot p\pi \qquad 0<p<1$

18.98 $\displaystyle\int_0^1 \frac{x^m - x^n}{\ln x}\,dx = \ln\frac{m+1}{n+1}$

18.99 $\displaystyle\int_0^\infty e^{-x}\ln x\,dx = -\gamma$

18.100 $\displaystyle\int_0^\infty e^{-x^2}\ln x\,dx = -\frac{\sqrt{\pi}}{4}(\gamma + 2\ln 2)$

18.101 $\displaystyle\int_0^\infty \ln\left(\frac{e^x+1}{e^x-1}\right)dx = \frac{\pi^2}{4}$

18.102 $\displaystyle\int_0^{\pi/2} \ln\sin x\,dx = \int_0^{\pi/2} \ln\cos x\,dx = -\frac{\pi}{2}\ln 2$

18.103 $\displaystyle\int_0^{\pi/2} (\ln\sin x)^2\,dx = \int_0^{\pi/2} (\ln\cos x)^2\,dx = \frac{\pi}{2}(\ln 2)^2 + \frac{\pi^3}{24}$

18.104 $\displaystyle\int_0^{\pi} x\ln\sin x\,dx = -\frac{\pi^2}{2}\ln 2$

18.105 $\displaystyle\int_0^{\pi/2} \sin x\ln\sin x\,dx = \ln 2 - 1$

18.106 $\displaystyle\int_0^{2\pi} \ln(a + b\sin x)\,dx = \int_0^{2\pi} \ln(a + b\cos x)\,dx = 2\pi\ln(a + \sqrt{a^2-b^2})$

18.107 $\displaystyle\int_0^{\pi} \ln(a + b\cos x)\,dx = \pi\ln\left(\frac{a+\sqrt{a^2-b^2}}{2}\right)$

18.108 $\displaystyle\int_0^{\pi} \ln(a^2 - 2ab\cos x + b^2)\,dx = \begin{cases} 2\pi\ln a, & a \geqq b > 0 \\ 2\pi\ln b, & b \geqq a > 0 \end{cases}$

18.109 $\displaystyle\int_0^{\pi/4} \ln(1+\tan x)\,dx = \frac{\pi}{8}\ln 2$

18.110 $\displaystyle\int_0^{\pi/2} \sec x\ln\left(\frac{1+b\cos x}{1+a\cos x}\right)dx = \tfrac{1}{2}\{(\cos^{-1}a)^2 - (\cos^{-1}b)^2\}$

18.111 $\displaystyle\int_0^{a} \ln\left(2\sin\frac{x}{2}\right)dx = -\left(\frac{\sin a}{1^2} + \frac{\sin 2a}{2^2} + \frac{\sin 3a}{3^2} + \cdots\right)$

See also 18.102.

DEFINITE INTEGRALS INVOLVING HYPERBOLIC FUNCTIONS

18.112 $$\int_0^\infty \frac{\sin ax}{\sinh bx}\,dx = \frac{\pi}{2b}\tanh\frac{a\pi}{2b}$$

18.113 $$\int_0^\infty \frac{\cos ax}{\cosh bx}\,dx = \frac{\pi}{2b}\operatorname{sech}\frac{a\pi}{2b}$$

18.114 $$\int_0^\infty \frac{x\,dx}{\sinh ax} = \frac{\pi^2}{4a^2}$$

18.115 $$\int_0^\infty \frac{x^n\,dx}{\sinh ax} = \frac{2^{n+1}-1}{2^n a^{n+1}}\Gamma(n+1)\left\{\frac{1}{1^{n+1}}+\frac{1}{2^{n+1}}+\frac{1}{3^{n+1}}+\cdots\right\}$$

If n is an odd positive integer, the series can be summed [see 19.19 to 19.21 and 23.8].

18.116 $$\int_0^\infty \frac{\sinh ax}{e^{bx}+1}\,dx = \frac{\pi}{2b}\csc\frac{a\pi}{b}-\frac{1}{2a}$$

18.117 $$\int_0^\infty \frac{\sinh ax}{e^{bx}-1}\,dx = \frac{1}{2a}-\frac{\pi}{2b}\cot\frac{a\pi}{b}$$

MISCELLANEOUS DEFINITE INTEGRALS

18.118 $$\int_0^\infty \frac{f(ax)-f(bx)}{x}\,dx = \{f(0)-f(\infty)\}\ln\frac{b}{a}$$

This is called *Frullani's integral*. It holds if $f'(x)$ is continuous and $\int_1^\infty \frac{f(x)-f(\infty)}{x}\,dx$ converges.

18.119 $$\int_0^1 \frac{dx}{x^x} = \frac{1}{1^1}+\frac{1}{2^2}+\frac{1}{3^3}+\cdots$$

18.120 $$\int_{-a}^{a} (a+x)^{m-1}(a-x)^{n-1}\,dx = (2a)^{m+n-1}\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$