Section X: Inequalities and Infinite Products

37 INEQUALITIES

TRIANGLE INEQUALITY

37.1 $$||a_1| - |a_2|| \leqq |a_1 + a_2| \leqq |a_1| + |a_2|$$

37.2 $$|a_1 + a_2 + \cdots + a_n| \leqq |a_1| + |a_2| + \cdots + |a_n|$$

CAUCHY-SCHWARZ INEQUALITY

37.3 $$(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \leqq (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2)$$

The equality holds if and only if $a_1/b_1 = a_2/b_2 = \cdots = a_n/b_n$.

INEQUALITIES INVOLVING ARITHMETIC, GEOMETRIC AND HARMONIC MEANS

If A, G and H are the arithmetic, geometric and harmonic means of the positive numbers $a_1, a_2, \ldots, a_n$, then

37.4 $$H \leqq G \leqq A$$

where

37.5 $$A = \frac{a_1 + a_2 + \cdots + a_n}{n}$$

37.6 $$G = \sqrt[n]{a_1 a_2 \ldots a_n}$$

37.7 $$\frac{1}{H} = \frac{1}{n}\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right)$$

The equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

HOLDER'S INEQUALITY

37.8 $$|a_1b_1 + a_2b_2 + \cdots + a_nb_n| \leqq (|a_1|^p + |a_2|^p + \cdots + |a_n|^p)^{1/p}(|b_1|^q + |b_2|^q + \cdots + |b_n|^q)^{1/q}$$

where

37.9 $$\frac{1}{p} + \frac{1}{q} = 1 \qquad p > 1,\ q > 1$$

The equality holds if and only if $|a_1|^{p-1}/|b_1| = |a_2|^{p-1}/|b_2| = \cdots = |a_n|^{p-1}/|b_n|$. For $p = q = 2$ it reduces to 37.3.

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CHEBYSHEV'S INEQUALITY

If $a_1 \geqq a_2 \geqq \cdots \geqq a_n$ and $b_1 \geqq b_2 \geqq \cdots \geqq b_n$, then

37.10 $$\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)\left(\frac{b_1 + b_2 + \cdots + b_n}{n}\right) \leqq \frac{a_1 b_1 + a_2 b_2 + \cdots + a_n b_n}{n}$$

or

37.11 $$(a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n) \leqq n(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)$$

[redacted]

If $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ are all positive and $p > 1$, then

37.12 $$\{(a_1 + b_1)^p + (a_2 + b_2)^p + \cdots + (a_n + b_n)^p\}^{1/p} \leqq (a_1^p + a_2^p + \cdots + a_n^p)^{1/p} + (b_1^p + b_2^p + \cdots + b_n^p)^{1/p}$$

The equality holds if and only if $a_1/b_1 = a_2/b_2 = \cdots = a_n/b_n$.

[redacted]

37.13 $$\left[\int_a^b f(x)g(x)\,dx\right]^2 \leqq \left\{\int_a^b [f(x)]^2\,dx\right\}\left\{\int_a^b [g(x)]^2\,dx\right\}$$

The equality holds if and only if $f(x)/g(x)$ is a constant.

[redacted]

37.14 $$\int_a^b |f(x)g(x)|\,dx \leqq \left\{\int_a^b |f(x)|^p\,dx\right\}^{1/p}\left\{\int_a^b |g(x)|^q\,dx\right\}^{1/q}$$

where $1/p + 1/q = 1$, $p > 1$, $q > 1$. If $p = q = 2$, this reduces to 37.13.

The equality holds if and only if $|f(x)|^{p-1}/|g(x)|$ is a constant.

[redacted]

If $p > 1$,

37.15 $$\left\{\int_a^b |f(x) + g(x)|^p\,dx\right\}^{1/p} \leqq \left\{\int_a^b |f(x)|^p\,dx\right\}^{1/p} + \left\{\int_a^b |g(x)|^p\,dx\right\}^{1/p}$$

The equality holds if and only if $f(x)/g(x)$ is a constant.

38 INFINITE PRODUCTS

38.1 $\sin x = x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\cdots$

38.2 $\cos x = \left(1-\frac{4x^2}{\pi^2}\right)\left(1-\frac{4x^2}{9\pi^2}\right)\left(1-\frac{4x^2}{25\pi^2}\right)\cdots$

38.3 $\sinh x = x\left(1+\frac{x^2}{\pi^2}\right)\left(1+\frac{x^2}{4\pi^2}\right)\left(1+\frac{x^2}{9\pi^2}\right)\cdots$

38.4 $\cosh x = \left(1+\frac{4x^2}{\pi^2}\right)\left(1+\frac{4x^2}{9\pi^2}\right)\left(1+\frac{4x^2}{25\pi^2}\right)\cdots$

38.5 $\frac{1}{\Gamma(x)} = xe^{\gamma x}\left\{\left(1+\frac{x}{1}\right)e^{-x}\right\}\left\{\left(1+\frac{x}{2}\right)e^{-x/2}\right\}\left\{\left(1+\frac{x}{3}\right)e^{-x/3}\right\}\cdots$

See also 25.11.

38.6 $J_0(x) = \left(1-\frac{x^2}{\lambda_1^2}\right)\left(1-\frac{x^2}{\lambda_2^2}\right)\left(1-\frac{x^2}{\lambda_3^2}\right)\cdots$

where $\lambda_1, \lambda_2, \lambda_3, \ldots$ are the positive roots of $J_0(x) = 0$.

38.7 $J_1(x) = x\left(1-\frac{x^2}{\lambda_1^2}\right)\left(1-\frac{x^2}{\lambda_2^2}\right)\left(1-\frac{x^2}{\lambda_3^2}\right)\cdots$

where $\lambda_1, \lambda_2, \lambda_3, \ldots$ are the positive roots of $J_1(x) = 0$.

38.8 $\frac{\sin x}{x} = \cos\frac{x}{2}\cos\frac{x}{4}\cos\frac{x}{8}\cos\frac{x}{16}\cdots$

38.9 $\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots$

This is called *Wallis' product.*

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